INTRODUCTION

TO

PLANE TRIGONOMETRY.

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PLANE TRIGONOMETRY

BY THE

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THIS little book consists partly of notes on Trigonometry which I have given at different times to my pupils, partly of bookwork which must be common to all works on the subject.

In all public schools but a few hours a week can be given to mathematics by the generality of boys, and of those few a very limited number can be assigned to the study of Trigonometry, yet the subject is one which should be read at school by all boys, and a knowledge of which is absolutely necessary for those who try for classical honours at Cambridge.

I have endeavoured to keep their requirements in my mind, and hope that this little book will be useful to sixth form classical boys, and that it may be a starting-point for younger boys who have some little taste for mathematics. While I trust that the bookwork is sufficiently clear and thorough, I have studied conciseness in writing it, and have not attempted the impossible task of making a boy independent of his master.

Many of the examples are original, others have been taken from recent papers set for the Previous Examination at Cambridge, or for entrance examinations for the army.

A few still more recent papers have been added.

CHARTERHOUSE, *December*, 1881.

CONTENTS.

ERRATA.

Before using this book, the reader is requested to make the following corrections.

Page 3, line 20, *for* 50^0 *read* 51^0.

„ 28, line 4 from bottom, *for* $\frac{PQ}{AP} = \cos B$, *read* $\frac{PQ}{AP} = \sin B$.

„ 30, line 12, *for* B for A, *read* A for B.

„ 58, question 1, *for* $\log_{12} 8$ *read* $\log_{\sqrt{2}} 8$.

„ 59, line 5 from bottom, *for* log 70·63700 *read* log 70·63600.

„ 62, question 16, *for* L sin 36^0 48′ 37″ *read* L sin 38^0 48′ 37″.

„ 70, question 32, *for* 23^0 *read* 33^0.

„ 74, line 7, *for* $\frac{280^0}{\pi}$ *read* $\frac{180^0}{\pi}$.

„ 77, question 14, *for* $\frac{n}{2}$ *read* $\frac{n}{4}$.

„ 82, question 19, *for* $2 \sin\theta -$ *read* $2 \sin\theta +$.

„ 88, question 17, *for* log 1·95 = ·292035, *read* log 1·95 = ·290035.

TRIGONOMETRY.

CHAPTER I.

MEASUREMENT OF ANGLES.

1. TRIGONOMETRY is the science which relates to the measurement of the sides and angles of triangles, and of angles generally. Its principal applications are in surveying, navigation, and astronomy.

A quantity or magnitude is said to be measured when the number of times it contains a fixed quantity is known.

Fixed magnitudes, by which other magnitudes of the same kind are measured, are called UNITS.

Thus the unit of length may be the foot, yard, inch, or mile.

We shall denote lengths of lines by letters such as a, b, x, y, meaning a units of length, or by two capital letters, such as AB.

Thus we may use the fraction $\frac{AB}{CD}$ to represent the numerical fraction whose numerator is the number of units of length in AB, and denominator the number of the same units in CD.

This fraction of course represents the ratio of the line AB to the line CD.

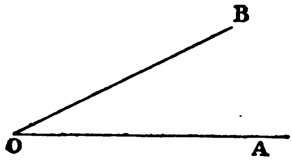
The natural unit of angular measurement is the right angle; this is however inconveniently large, so it is divided into 90 equal parts, called degrees, each of these into 60 others called minutes, and each minute into 60 subdivisions called seconds.

$60^\circ\ 54'\ 30''$ means 60 degrees, 54 minutes, 30 seconds. Angles are denoted by capital letters as A, B, or by Greek letters such as α, β, θ, but not by ordinary small letters which denote the lengths of lines.

2. *Angles may be of any magnitude.*

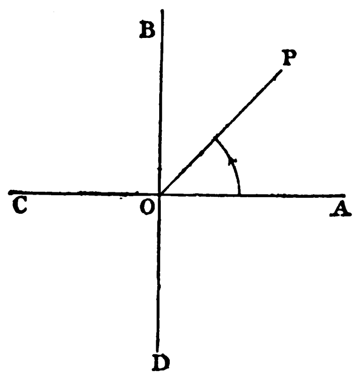
We shall *always* suppose an angle to be formed by the revolution of one of its arms from coincidence with the other.

Thus to form the angle AOB, OB must be supposed to have revolved from OA to OB.



Now it is obvious that OB may go on revolving until it comes into the same position as many times as we please, AOB may therefore be greater than 2, 4, or any number of right angles.

Thus between 2 and 4 o'clock the minute hand of a watch has revolved through 8 right angles or 720°.



Through any point O draw the straight lines AOC, BOD at right angles to each other.

Let OP revolve from OA in the direction *opposite to that of the revolution of watch hands.*

Then it is said to form the angle AOP.

If $AOP < 1$ right angle AOP is said to be in the first quadrant,

if $AOP > 1$ right angle, but < 2 it is said to be in the second quadrant,

if $\quad > 2$ but < 3 right angles, in the third,

if $\quad > 3$ but < 4 right angles, in the fourth.

OP, the line which by its revolution generates the angle, is called the generating line.

3. It has been proposed to divide a right angle into 100 equal parts called grades, each grade into 100 minutes, each minute into 100 seconds. An angle thus expressed would be written $60^{g}\ 54^{`}\ 30^{``}$. This system is called the French or *centesimal* system, as opposed to the ordinary or *sexagesimal* system.

If D be the number of degrees in any angle, G that of grades, $\frac{D}{90} = \frac{G}{100}$, since each fraction expresses the ratio of the angle to a right angle. Thus to reduce degrees, &c. to grades express the angle in degrees and decimals of a degree, multiply by 10 and divide by 9.

Ex. Express $50^{\circ}\ 4'\ 30''$ in centesimal measure.

60 | 30	9 | 510·75
60 | 4·5	56·75
51·075	

$\therefore\ 51^{\circ}\ 4'\ 30'' = 56^{g}\ 75^{`}\ 0^{``}$.

For proof reverse the process.

Express $87^{g}\ 2^{`}\ 25^{``}$ in degrees &c.

$$\begin{array}{rl} 87^{g}\ 2^{`}\ 25^{``} = & 87{\cdot}0225^{g} \\ & \underline{\qquad\quad {\cdot}9} \\ & 78{\cdot}32025 \\ & \underline{\qquad\quad 60} \\ & \quad 19{\cdot}215 \\ & \underline{\qquad\quad 60} \\ & \quad 12{\cdot}9 \end{array}$$

Here we multiply by ·9 since each grade is $\frac{9}{10}$ of a degree, and we write 13″ in the answer because 12·9 is nearer 13 than 12.

Ans. $78^{\circ}\ 19'\ 13''$.

EXAMPLES ON CHAPTER I.

1. Transform 60^0, $57^0\,30'$, $45^0\,33'\,3''$ to grades, &c.

2. Transform $65^g\,31^`\,25^{``}$, $174^g\,81^`\,3^{``}$, $5^g\,5^`\,5^{``}$ to degrees, &c.

3. Through how many degrees do the hands of a watch revolve between 4 h. and 6 h. 25 min.?

4. It is between 4 and 5 p.m. and the hands of a watch are 100^0 apart, what is the time?

5. Express in degrees the angles of the triangle in *Euc.* iv. 10.

6. If the unit angle be x degrees, what is the measure of an angle of an equilateral triangle?

7. If the unit angle be x grades, what is the measure of b degrees?

8. A degree measures x units, how many grades measure y units?

9. If the sum of the angles of a triangle be equal to the unit, what is the number of degrees in the unit?

10. What must be the unit angle, if the sum of the measures of a degree and a grade be 5.7?

11. Find the number of degrees in an angle of a regular pentagon, also of a hexagon and a decagon.

12. If n be the number of sides of a regular polygon, the number of degrees in each angle is $\frac{n-2}{n} \times 180$.

13. An angle of a regular figure is 170^0, how many sides has it?

14. The number of sides of one regular polygon is double that of another, and the ratio of an angle of the first to one of the second is 5 : 4; find the number of sides in each.

15. The number of sides of one regular polygon is $\frac{5}{4}$ that of another, and the difference between the angles is 9^0; find the number of sides in each.

16. The number of sides of one regular polygon : number of another :: $\angle$ of first : $\angle$ of second; determine the number of sides and the angles of each.

17. The angles of a triangle are in A.P. and the greatest is treble the least; find them.

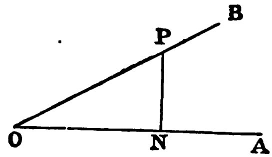
18. $ABCD$ is a quadrilateral in a circle: A is 45^0, B 75^0, what are C and D?

Note. Chapter XVI., Arts. 65–70, on Circular Measure may be read now, if it be thought desirable to do so.

CHAPTER II.

TRIGONOMETRICAL RATIOS.

4. LET AOB be an acute angle, A its measure in degrees. Take any point P in OB, and draw PN perpendicular to OA, thus forming a triangle whose shape is fixed if A is fixed.



The ratios of the sides of this triangle are called the ratios of the angle A, and have received names which must be committed to memory.

They are six in number, since each side can be compared with two others. Let us call PN, the side opposite to O, the perpendicular, ON the base, OP the hypotenuse.

Then

$\frac{PN}{OP} = \frac{\text{perp}}{\text{hyp}}$ is called the sine of AOB, and written sin A.

$\frac{ON}{OP} = \frac{\text{base}}{\text{hyp}}$ .... cosine .... cos A.

$\frac{PN}{ON} = \frac{\text{perp}}{\text{base}}$ .... tangent .... tan A.

$\frac{ON}{PN} = \frac{\text{base}}{\text{perp}}$ .... cotangent .... cot A.

$\frac{OP}{ON} = \frac{\text{hyp}}{\text{base}}$ .... secant .... sec A.

$\frac{OP}{PN} = \frac{\text{hyp}}{\text{perp}}$ .... cosecant .... cosec A.

$1 - \cos A$ is called the versed sine and is written vers A,

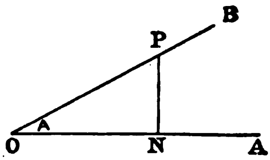
$1 - \sin A$ .... coversed sine .... covers A.

5. If an angle be taken from a right angle, the remainder is called its *complement*, if from two right angles, its *supplement*.

Thus $90^\circ - A$, $180^\circ - A$ are the complement and supplement of A.

In the figure OPN is the complement of AOB.

Now if we form the ratios of OPN, by taking O in PO, and drawing ON perpendicular to PN, we get the same triangle as before, but the base and perpendicular are interchanged, the hypotenuse remaining unaltered.



Hence $\sin(90^\circ - A) = \dfrac{ON}{OP} = \cos A$, $\cos(90^\circ - A) = \sin A$,

$$\tan(90^\circ - A) = \cot A, \quad \cot(90^\circ - A) = \tan A,$$
$$\sec(90^\circ - A) = \operatorname{cosec} A, \quad \operatorname{cosec}(90^\circ - A) = \sec A.$$

Hence we see, that by adding or taking off the prefix *co*, the ratios of any angle become those of its complement.

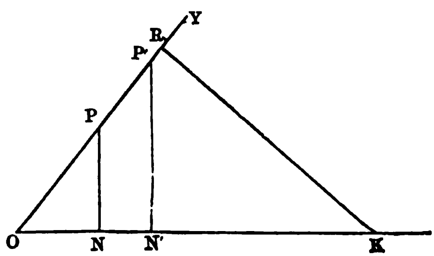
6. Acute angles which have the same ratios are equal.

For if two angles have their tangents or cotangents equal, then we have two right-angled triangles having the sides about the right angles proportional, and if any other ratios be equal, these triangles have the sides about some other angles proportional, ∴ in each case the triangles are equiangular.

7. *The trigonometrical ratios remain unchanged so long as the angle does so.*

Let AOB be any acute angle, take P, P', any points in OB, and draw PN, $P'N'$ perpendicular to OA.

Take K any point in OA, and draw KR perpendicular to OB.



Then the triangles PON, $P'ON'$, KOR are equiangular, since they are right-angled, and have a common angle at O.

Therefore the sides about the equal angles are proportionals, that is, $\frac{PN}{OP} = \frac{P'N'}{OP'} = \frac{KR}{OK}$, and similarly for the other ratios.

Hence we see that to every angle a special set of ratios belongs, and conversely, given any ratio, such as the sine, there is only one acute angle of which it is the sine.

Again, since $OP^2 = ON^2 + PN^2$, if two of these quantities be given we know the third, and if the ratio of any two be given, we know that of the third to either.

All the ratios of any angle must therefore be capable of being found when one is given, and this will be the object of Chapter III.

EXAMPLES ON CHAPTER II.

1. ABC is a triangle, C being a right angle; from C CD is drawn perpendicular to AB; write down in as many ways as you can the ratios of the angles at A and B.

2. The supplement of an angle is treble the complement: how many degrees are there in the angle?

3. The supplement of the complement of an angle is 120°; what is the angle?

4. The complement of the supplement of an angle is 18°; what is the angle?

5. O is the centre of a circle ABD; TA, TB tangents at A, B; if the angle AOB be A, find the angles of the triangle TAB.

6. If the triangle TAB in the preceding question be equilateral, find the angles of the triangle OAB.

7. ABC is a triangle, right-angled at C, if D be a point in CB such that $CD : DB :: \cos BAC : 1$, then BAC is bisected by AD.

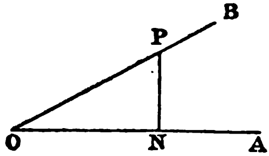
8. Shew that the sine and cosine of an angle can never be greater than unity, and the secant and cosecant never less.

9. Shew that if $\tan A$ be greater than $\cot A$, $A > 45^\circ$.

CHAPTER III.

RELATIONS BETWEEN THE RATIOS.

8. LET AOB be an acute angle, whose measure in degrees is A, then if we know one of its ratios we can find all the others.



We will find equations connecting these ratios, shew how to find one when another is given, and how to express any one in terms of any other.

Forming the triangle as before,

$$\text{since } \sin A = \frac{PN}{OP}, \text{ and } \operatorname{cosec} A = \frac{OP}{PN};$$

$$\therefore\ \sin A \,.\, \operatorname{cosec} A = 1 \text{ or } \operatorname{cosec} A = \frac{1}{\sin A} \quad\text{............(i).}$$

Just in the same way we can prove that

$$\cos A \,.\, \sec A = 1 \quad\text{.....................(ii),}$$

and that

$$\tan A \,.\, \cot A = 1 \quad\text{....................(iii).}$$

Hence, having given one of either of these sets we can find the other of that set.

Again,

$$\tan A = \frac{PN}{ON} = \frac{\dfrac{PN}{OP}}{\dfrac{ON}{OP}} = \frac{\sin A}{\cos A} \quad\text{................. (iv).}$$

This relation may be written in either of the forms

$$\cot A = \frac{\cos A}{\sin A}, \text{ or } \tan A \cos A = \sin A.$$

Again, $PN^2 + ON^2 = OP^2$. *Euc.* i. 47.

Divide by each of these quantities in turn,

$$\left.\begin{array}{ll} \therefore \text{ either } & \dfrac{PN^2}{OP^2} + \dfrac{ON^2}{OP^2} = 1; \text{ that is, } \sin^2 A + \cos^2 A = 1; \\ \text{or} & \dfrac{PN^2}{ON^2} + 1 = \dfrac{OP^2}{ON^2}, \tan^2 A + 1 = \sec^2 A; \\ \text{or} & 1 + \dfrac{ON^2}{NP^2} = \dfrac{OP^2}{PN^2}, \cot^2 A + 1 = \operatorname{cosec}^2 A. \end{array}\right\} \text{(v).}$$

These three equations are not independent, being only different ways of expressing the relation $PN^2 + ON^2 = OP^2$.

We have now five equations between six quantities and can therefore find the values of all the others if one be given.

Observe that

$$\cos A = \sqrt{1 - \sin^2 A}, \ \sin A = \sqrt{1 - \cos^2 A},$$
$$\sec A = \sqrt{1 + \tan^2 A}, \operatorname{cosec} A = \sqrt{1 + \cot^2 A}.$$

We take the positive sign of the root in each case; the meaning of taking the negative sign will be explained hereafter.

9. Given any ratio, to find the others.

(i) *First Method.*

Let PON be a right-angled triangle, and suppose PON to be the angle, one of whose ratios is given.

Take that side which denotes the antecedent of the given ratio, and divide it into as many equal parts as are shewn by the numerator.

The side which represents the consequent will therefore contain as many such parts as are denoted by the denominator.

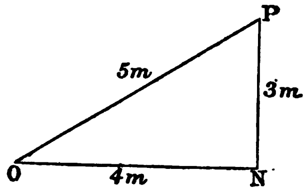
Then the relation $OP^2 = ON^2 + PN^2$ gives the third side. Hence we can write down any assigned ratio.

For instance, given

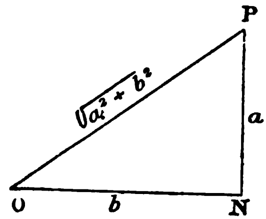
$\sin A = \frac{3}{5}$, find $\tan A$.

If $PN = 3m$, $OP = 5m$,

$\therefore$ $ON = 4m$, and $\tan A = \frac{3}{4}$.



If $\tan A = \dfrac{a}{b}$, and PN be divided into a equal parts, ON will contain b such parts; $\therefore$ OP will contain $\sqrt{a^2 + b^2}$ such parts.



$$\therefore \sin A = \frac{a}{\sqrt{a^2 + b^2}},$$

$$\cos A = \frac{b}{\sqrt{a^2 + b^2}}.$$

(ii) *Second Method.*

To express all the ratios in terms of one, the sine for instance.

Since $\sin^2 A + \cos^2 A = 1$, $\cos A = \sqrt{1 - \sin^2 A}$.

$$\therefore \tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}, \quad \cot A = \frac{1}{\tan A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}}, \quad \operatorname{cosec} A = \frac{1}{\sin A}.$$

Observe that $\sin A$ and $\cos A$ are necessarily < 1 since OP is the greatest side of the triangle, and that $\sec A$ and $\operatorname{cosec} A$ are > 1 for the same reason.

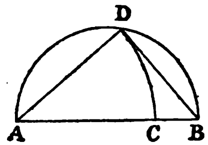
Tan A and $\cot A$ may be of any magnitude.

10. Given any ratio, to find the angle.

(i) Let $\sin A = \frac{a}{b}$ or $\cos A = \frac{a}{b}$, or let $\sec A = \frac{b}{a}$ or $\operatorname{cosec} A = \frac{b}{a}$, where $b > a$.

Take AB b units of length and on it describe a semicircle.

Take AC a units of length, and with centre A, radius AC, describe a circle cutting the former circle in D: join AD, BD.



Then ADB is a right angle (*Euc.* iii. 31) and $AD = a$;

$\therefore$ the angle DBA has its sine $\frac{a}{b}$ and cosecant $\frac{b}{a}$, and the angle DAB has its cosine $\frac{a}{b}$, and its secant $\frac{b}{a}$.

(ii) Let $\tan A = \frac{a}{b}$ or $\cot A = \frac{b}{a}$.

Draw AB equal to b and BC perpendicular to AB equal to a: join CA.

Then $\tan CAB = \frac{a}{b}$ and $\cot CAB = \frac{b}{a}$.

CAB is $\therefore$ the angle required.

EXAMPLES ON CHAPTER III.

1. Given $\tan A = \frac{4}{3}$, find all the other ratios.
2. Given $\sin A = \frac{12}{13}$, find all the other ratios.
3. If $\sec A = \frac{41}{40}$, find $\sin A$ and $\cot A$.
4. If $\operatorname{vers} A = \frac{1}{13}$, find $\tan A$.
5. If $\tan A = \frac{11}{60}$, find $\operatorname{vers} A$ and $\sin A$.
6. If $\tan A = 3\frac{3}{7}$, find $\sec A$.
7. Express the other ratios in terms of the cosine and tangent.

Prove the following:

8. $\operatorname{cosec}\theta \tan\theta \cos\theta = \sec\theta \cot\theta \sin\theta = 1.$

9. $\sec A = \dfrac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A - 1}}.$

10. $\tan A = \dfrac{\sec A}{\sqrt{1+\cot^2 A}}.$

11. $\dfrac{\sqrt{\sec^2 A - 1}}{\sqrt{\operatorname{cosec}^2 A - 1}} = \tan^2 A.$

12. $\sin^2 A + \cos^2 A + \tan^2 A + \cot^2 A + 1 = \sec^2 A + \operatorname{cosec}^2 A.$

13. $\dfrac{1}{1+\tan^2 A} + \dfrac{1}{1+\cot^2 A} = 1.$

14. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x.$

15. $\sin^4 \alpha + \cos^4 \alpha = 1 - 2\sin^2\alpha \cos^2\alpha.$

16. $\cos^6 \alpha + \sin^6 \alpha = 1 - 3\sin^2\alpha\cos^2\alpha.$

17. $(\tan A + \cot A)^2 = \sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A.$

18. $(\sin\alpha + \cos\alpha)(\tan\alpha + \cot\alpha) = \sec\alpha + \operatorname{cosec}\alpha.$

19. $\sin^2 A \cos^2 B\,(1+\cot^2 A)(1+\tan^2 B) = 1.$

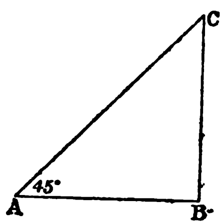
20. $(\sec^2\alpha - \cos^2\alpha)(\operatorname{cosec}^2\alpha - \sin^2\alpha) = 2 + \sin^2\alpha\cos^2\alpha.$

CHAPTER IV.

RATIOS OF CERTAIN ANGLES.

11. *To find the ratios of* 45°.

Take any straight line AB and draw BC perpendicular to AB and equal to it.



Join CA; then each of the angles at A and C is half a right angle or 45°.

Now $AC^2 = AB^2 + BC^2 = 2AB^2$,

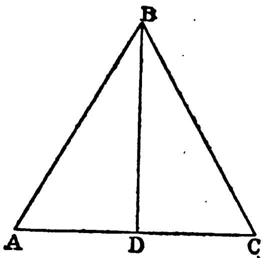
$\therefore AC = \sqrt{2}AB.$

$$\therefore \sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}, \quad \cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}:$$

so $\tan 45^\circ = \dfrac{BC}{AC} = 1 = \cot 45^\circ$, $\operatorname{cosec} 45^\circ = \sqrt{2} = \sec 45^\circ$. AB

12. *To find the ratios of* 60° *and* 30°.

Describe an equilateral triangle ABC; bisect AC in D; join BD. Then obviously BD is perpendicular to AC, and BD bisects ABC.



Now since all the angles of a triangle are equal to two right angles;

$\therefore BAD = 60^\circ$, and $ABD = 30^\circ$.

Also $BD^2 = AB^2 - AD^2 = 3AD^2$, $\therefore BD = \sqrt{3}AD$.

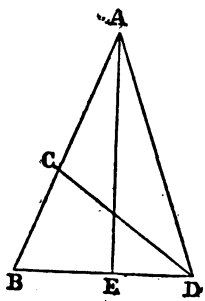
$$\therefore \sin 60^\circ = \frac{BD}{AB} = \frac{\sqrt{3}}{2} = \cos 30^\circ, \quad \cos 60^\circ = \frac{AD}{AB} = \frac{1}{2} = \sin 30^\circ,$$

$$\tan 60^\circ = \frac{BD}{AD} = \sqrt{3} = \cot 30^\circ, \quad \cot 60^\circ = \frac{AD}{BD} = \frac{1}{\sqrt{3}} = \tan 30^\circ,$$

$$\sec 60^\circ = \frac{AB}{AD} = 2 = \operatorname{cosec} 30^\circ, \quad \operatorname{cosec} 60^\circ = \frac{AB}{BD} = \frac{2}{\sqrt{3}} = \sec 30^\circ.$$

13. *To find the ratios of* 18°.

Construct (by *Euc.* IV. 10) the triangle ABD, having each of the angles at B and D double the angle at A.



Bisect the angle BAD by AE cutting BD at E. Then obviously the angles at E are right angles.

Take AC equal to BD, then it is proved by Euclid that $AC^2 = AB \cdot BC$, and that $AC = BD$.

Let $BD = x$, $AB = a$, $\therefore x^2 = a(a - x)$.

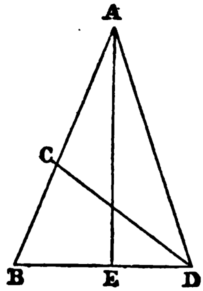
Whence, taking the positive root of this equation,

$$x = \frac{\sqrt{5}-1}{2} a.$$

Now since BAD is $\frac{1}{5}$ the sum of the angles of the triangle BAD;

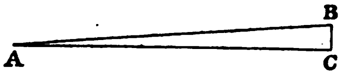
$$\therefore BAD = 36^\circ, \; BAE = 18^\circ,$$

$$\therefore \sin 18^\circ = \frac{x}{2a} = \frac{\sqrt{5}-1}{4}.$$



The other ratios may be found from the sine.

14. *To find the ratios of* 0° *and* 90°.



Let BAC be a right-angled triangle, A a very small angle, and therefore B nearly a right angle.

Then the smaller A is, the smaller is BC in comparison with AB, and the more nearly do AB and AC become equal.

Now by sufficiently diminishing A we can make BC less than any finite magnitude, and therefore $AB^2 - AC^2$ may be made less than any finite magnitude.

That is to say, the more we diminish A the more nearly $BC = 0$, $AC = AB$.

$$\therefore \sin 0^\circ = \frac{BC}{AB} = 0, \; \cos 0^\circ = 1, \; \tan 0^\circ = 0,$$

$$\cot 0^\circ = \infty, \; \sec 0^\circ = 1, \; \operatorname{cosec} 0^\circ = \infty.$$

$$\sin 90^\circ = 1, \; \cos 90^\circ = 0, \; \tan 90^\circ = \infty, \; \cot 90^\circ = 0,$$

$$\sec 90^\circ = \infty, \; \operatorname{cosec} 90^\circ = 1.$$

These must be considered *abbreviated statements* for the following.

When an angle is very small, its sine is very small, and by diminishing the angle its sine can be made less than any assignable quantity, and so can its tangent.

So too its cotangent can be made greater than any assignable quantity, and its cosine nearly equal to unity.

Again, when an angle is very nearly a right angle, its tangent is very large, and the more nearly it approaches a right angle, the more we increase its tangent and diminish its cosine.

∞ is the symbol for *infinity*, or a number greater than any assignable number.

EXAMPLES ON CHAPTER IV.

1. $\dfrac{\cos 60^0 + \cos 30^0}{\sec 60^0 + \operatorname{cosec} 60^0} = \dfrac{\sqrt{3}}{4}$.

2. $(\cos 30^0 + \cos 45^0)(\sec 30^0 - \tan 45^0) = \dfrac{\cot 60^0 - \sin 30^0}{\cot 30^0 - \sec 45^0}$.

3. $\sin 60^0 + \cos 60^0 + \tan 60^0 = \sin 30^0 + \cos 30^0 + \cot 30^0$.

4. $\sin^2 60^0 - \sin^2 30^0 = \frac{1}{3} \tan 60^0 \cot 30^0$.

5. $(\sin 30^0 + \cos 30^0)(\sin 60^0 - \cos 60^0) = \sin 30^0$.

6. $\dfrac{\cos 45^0 - \cos 60^0}{\sin 45^0 + \sin 30^0} = (\operatorname{cosec} 45^0 - \cot 45^0)^2$.

7. $\sin^2 30^0$, $\sin^2 45^0$, $\sin^2 60^0$, $\sin^2 90^0$ are in A.P.

8. A, B, C are the angles of an isosceles triangle, and $2 \sin A = \tan A$, find A, B, C.

9. ABC is a triangle, $\sin (A - B) = \frac{1}{2}$, $\cos C = 0$, find A, B, C.

10. Find $\cot 18^0$, $\tan 18^0$, $\operatorname{cosec} 18^0$, $\sec 72^0$.

11. $\sin (3A - 5B) = 0$, $\cos 2B = \dfrac{\sqrt{5} - 1}{4}$; find A and B.

12. $\cos A = \sqrt{2} \cos B$, $\tan B = \sqrt{3} \tan A$; find A and B.

13. $\sin A = \sqrt{2} \sin B$, $\sqrt{3} \cos A = \sqrt{2} \cos B$; find A and B.

14. If $\sin A + \operatorname{cosec} A = 2\frac{1}{2}$, find a value of A.

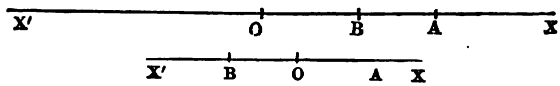
15. $3 \tan^2 A + \cot^2 A = 4$; find two values of A.

16. $\cot A = \tan 2A$; find a value of A.

CHAPTER V.

USE OF THE SIGNS + AND − TO SHEW DIRECTION.

15. LET $X'OAX$ be a straight line, O a fixed point in it, A a point to the right such that $OA = a$.

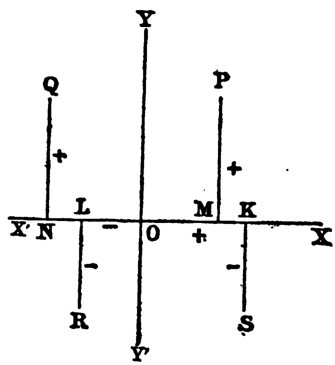


Measure AB towards O so that $AB = b$. Then $OB = a - b$.

Now if $a > b$, B lies to the right of O, but if $b > a$, B lies to the left. Therefore if $x = a - b$, B lies to the right or left of O, as $a >$ or $< b$, that is as x is positive or negative.

If then we consider a line drawn in one direction as positive, we must consider that drawn in the opposite direction negative. Such lines are sometimes said to differ in *sense*. Sometimes the direction in which a line is drawn is shewn by the order of the letters; thus AB means a line drawn from A to B, but BA one from B to A, and then $AB + BA = 0$.

Through O draw $Y'OY$ perpendicular to OX, and take points P, Q, R, S lying in the four quadrants thus formed.

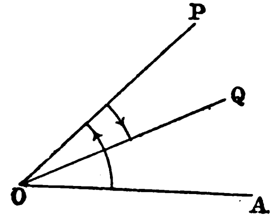


Draw PM, QN, RL, SK perpendicular to XO.

Then if we take OM, PM as positive, QN will also be positive, since it is drawn in the same direction as PM, but RL, SK will be negative.

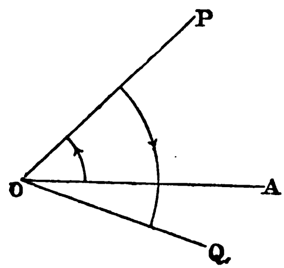
Thus if lines perpendicular to OX drawn from points above OX are positive, those drawn from points below will be negative.

16. *Positive and negative angles.*



Let the angle AOP be formed by OP revolving through A degrees from OA, and then let it revolve back through B degrees to OQ. Then $AOQ = A - B$ degrees.

Now if $A > B$ or $A - B$ be positive, OQ will be above OA, but if $A < B$ or $A - B$ be negative OQ will be below OA.



If then we consider B to be zero, we get an angle formed by OP revolving in one direction, but if we consider A to be zero, we get an angle formed by revolution in the opposite direction.

The signs + or − shew ∴ the direction in which the generating line revolves.

We consider an angle formed by the revolution of OP in the direction *opposite to that of watch-hands* to be positive.

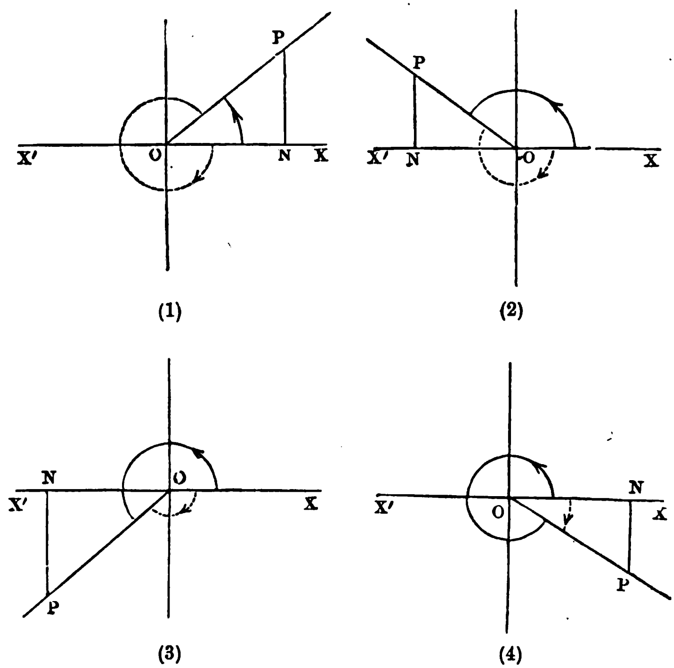
17. We can now give extended definitions of the trigonometrical ratios.

If from any point in one arm of an angle a perpendicular be drawn to the other arm, or that arm produced, thus forming a right-angled triangle, the ratios are those of its sides as previously defined, the sides including the right angle being positive or negative according to the direction in which they are drawn.

The hypotenuse or *generating line* is *always positive.*

The relations of Chap. III. between the ratios still subsist, since they are derived from the properties of triangles.

18. To form a table of the signs of the ratios in the 4 quadrants.



Let XOP be an angle, formed by OP revolving in the positive direction or, as shewn by the dotted lines, in the negative direction.

Draw PN perpendicular to OX or OX produced.

Then in the 1st quadrant all the ratios are positive.

In the 2nd ON is negative, PN and OP positive; therefore the sine and cosecant are positive, the rest negative.

In the 3rd ON and PN are both negative, OP positive; therefore the sine, cosine, secant, cosecant are negative, but the tangent and cotangent positive.

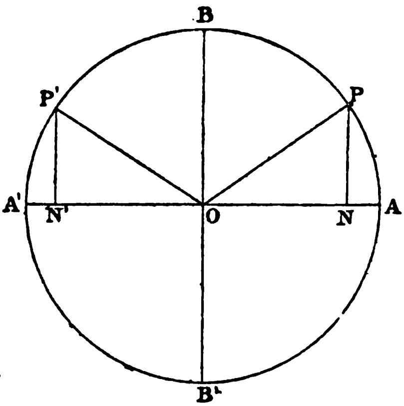
In the 4th ON is positive, PN negative, OP is positive; therefore the sine, tangent, cotangent and cosecant are negative. The annexed table shews these signs.

sin or cosec	+	+	−	−	
cos or sec	+	−	−	+	
tan or cot	+	−	+	−	

CHAPTER VI.

ANGLES WHOSE RATIOS DIFFER ONLY IN SIGN.

19. *To find the ratios of* $180° - A$ *in terms of those of* $A°$.



With centre O, radius OA, describe a circle.

Let AOA', BOB' be two diameters at right angles.

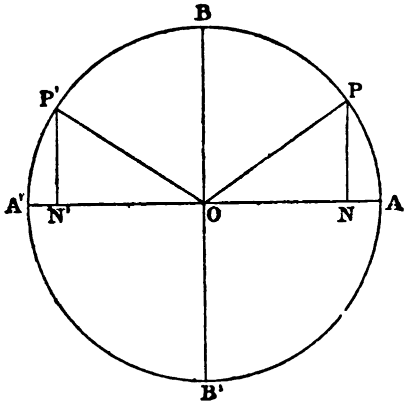
Let the generating line OP revolve from the position OA through $A°$, and let OP' revolve from OA' in the opposite direction through $A°$;

$\therefore AOP' = 180° - A°$.

Draw PN, $P'N'$ perpendicular to AOA'.

Then (by Euc. I. 26), the triangles PON, $P'ON'$ are equal in all respects. But since $A'OP' = AOP$, P and P' are always on the same side of AOA' and on opposite sides of

BOB'; that is $P'N'$ and PN are of the same sign, ON', ON of opposite signs; $\therefore P'N' = PN$, $ON' = -ON$, $OP' = OP$.



$\therefore \sin(180^\circ - A)$

$$= \frac{P'N'}{OP'} = \frac{PN}{OP} = \sin A;$$

and $\therefore \operatorname{cosec}(180^\circ - A) = \operatorname{cosec} A$.

$\operatorname{Cos}(180^\circ - A)$

$$= \frac{ON'}{OP'} = -\frac{ON}{OP} = -\cos A;$$

and

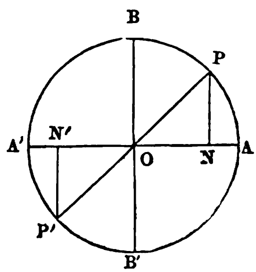
$\therefore \sec(180^\circ - A) = -\sec A$.

$\operatorname{Tan}(180^\circ - A)$

$$= \frac{P'N'}{ON'} = -\frac{PN}{ON} = -\tan A;$$

and $\therefore \cot(180^\circ - A) = -\cot A$.

20. *To find the ratios of* $180^\circ + A$ *in terms of those of* A.



Describe the circle as before, but now let OP' revolve in the same direction as OP and through the same angle A.

Therefore POP' is a straight line and $AOP' = 180^\circ + A$.

Here, as before, the triangles $P'ON'$, PON are geometrically equal in all respects. Now P and P' are on opposite sides of both AOB' and BOB';

$$\therefore P'N = -PN, \quad ON' = -ON, \quad OP' = OP;$$

$$\therefore \sin(180^\circ + A) = \frac{P'N'}{OP'} = -\frac{PN}{OP} = -\sin A:$$

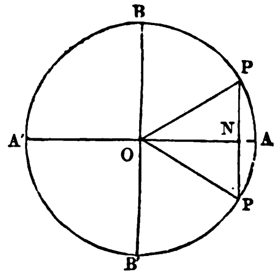
$$\cos(180^\circ + A) = \frac{ON'}{OP'} = -\frac{ON}{OP} = -\cos A:$$

$$\tan(180^\circ + A) = \frac{P'N'}{ON'} = \frac{-PN}{-ON} = \frac{PN}{ON} = \tan A.$$

The cosecant, secant, and cotangent have, of course, the same sign as their reciprocals.

21. *To find the ratios of* $-A$ *in terms of those of* A.

Describe the circle as before.



Let the two generating lines OP, OP' revolve from OA through the same angle A in opposite directions, then $AOP' = -A$.

Join PP' cutting OA or OA produced in N.

Then, since $OP = OP'$, the angles at N are right angles, PP' is bisected at N and the triangles PON, $P'ON$ are geometrically equal in all respects; and since P', P are on opposite sides of AA', $P'N = -PN$, $OP' = OP$.

$$\therefore \sin(-A) = \frac{P'N}{OP'} = -\frac{PN}{OP} = -\sin A;$$

$$\therefore \operatorname{cosec}(-A) = -\operatorname{cosec} A:$$

$$\cos(-A) = \frac{ON}{OP'} = \frac{ON}{OP} = \cos A; \quad \sec(-A) = \sec A:$$

$$\tan(-A) = \frac{P'N}{ON} = -\frac{PN}{ON} = -\tan A; \quad \cot(-A) = -\cot A.$$

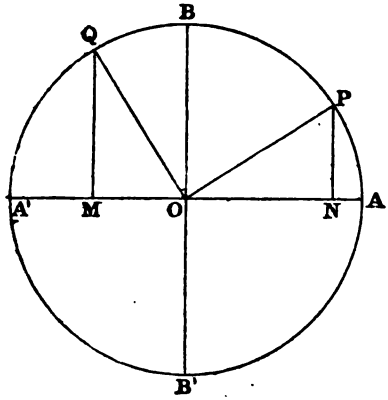
In all these proofs P may be anywhere on the circle; therefore AOP may be of any value.

We have seen that no two acute angles can have the same ratios, and exactly the same proof shews that no two angles in the same quadrant can have the same ratios.

Therefore the angles $180^\circ - A$, $180^\circ + A$, $-A$, and angles formed by adding or subtracting multiples of 360° to these, are the only angles whose ratios are numerically equal.

22. *To find the ratios of* $90^\circ + A$ *in terms of those of* A.

Let $AOP = A$ and draw OQ perpendicular to OP.



Describe the circle as before.

Draw PN, QM perpendicular to AA'.

Then $AOQ = 90^\circ + A$.

Now $MQO = QOB = PON$, and the angles at N, M, are right angles, also $OP = OQ$.

$\therefore$ the triangles QOM, OPN are geometrically equal in all respects.

Therefore $PN = OM$, $ON = QM$, geometrically.

Now as P is above or below AA', Q is to the left or right of BB'; therefore PN is positive if OM is negative, and negative if OM is positive: so, if P is to the right of BB', Q is above AA'; therefore QM and ON are of the same sign.

$$\therefore \frac{QM}{OQ} = \frac{ON}{OP}, \text{ or } \sin(90^\circ + A) = \cos A;$$

$$\frac{OM}{OQ} = -\frac{PN}{OP}, \text{ or } \cos(90^\circ - A) = -\sin A.$$

23. We may find the ratios of $90^\circ - A$, $270^\circ \pm A$ in the same way.

If OP make a complete revolution after describing A, it comes into the same position as before; therefore we do not alter the ratios of any angle by either adding or subtracting any multiple of 360°.

If then we know the ratios of all acute angles, we can find those of all angles positive or negative. It is enough if we know the ratios of angles up to 45°, as then we know those of their complements, which are between 45° and 90°.

Ex. Find $\sin 120^0$, $\cos 240^0$, $\tan 480^0$, $\cot(-600^0)$, $\sec(-7335^0)$, $\operatorname{cosec} 4122^0$.

$$\sin 120^0 = \sin(180^0 - 60^0) = \sin 60^0 = \frac{\sqrt{3}}{2}.$$

$$\cos 240^0 = \cos(180^0 + 60^0) = -\cos 60^0 = -\tfrac{1}{2}.$$

$$\tan 480^0 = \tan(360^0 + 120^0) = \tan 120^0 = \tan(180^0 - 60^0) = -\tan 60^0 = -\sqrt{3}.$$

$$\cot(-600^0) = -\cot 600^0 = -\cot(360^0 + 240^0) = -\cot 240^0$$
$$= -\cot(180^0 + 60^0) = -\cot 60^0 = -\frac{1}{\sqrt{3}}.$$

$$\sec(-7335^0) = \sec 7335^0 = \sec(360 \times 20 + 135^0) = \sec 135^0$$
$$= \sec(180^0 - 45^0) = -\sec 45^0 = -\sqrt{2}.$$

$$\operatorname{cosec} 4122^0 = \operatorname{cosec}(360^0 \times 11 + 162^0) = \operatorname{cosec} 162^0$$
$$= \operatorname{cosec}(180^0 - 18^0) = \operatorname{cosec} 18^0 = \frac{4}{\sqrt{5} - 1} = \sqrt{5} + 1.$$

EXAMPLES ON CHAPTER VI.

1. Shew that $\cos(90^0 - A) = \sin A$, $\cot(90^0 - A) = \tan A$, for all values of A.

2. Find the ratios of $270^0 + A$ and $270^0 - A$ in terms of those of A.

3. Find all the ratios of 135^0, 150^0, 240^0, 300^0.

4. Find the sines of -240^0, 405^0, -45^0, 750^0, 810^0, 4005^0.

5. Find the cosines of 510^0, -30^0, 3540^0, 225^0, 315^0, and the tangents of 120^0, 225^0, -585^0, 750^0, 7320^0.

6. Find $\sin 480^0$, $\cos 4080^0$, $\tan 8400^0$, $\cot -7260^0$, $\sec 7395^0$, $\operatorname{cosec} 1485^0$, $\operatorname{vers} 1500^0$.

7. Reduce $\sin 7321^0$, $\cos -8146^0$, $\tan 7389^0$, $\cot 375^0$, $\sec -8325^0$, $\operatorname{cosec} 1732^0$, $\operatorname{vers} 1818^0$, to ratios of angles less than 45^0.

8. If $\sin B = \sin A$, $\cos B = \cos A$, then $A - B$ is either 0 or a multiple of 360^0.

9. If $\cos B = \cos A$, $\tan B = -\tan A$, then $A + B = 0$ or a multiple of 360^0.

10. If $\sin A = \sin B$, $\cos A = -\cos B$, then $A + B$ is an odd multiple of 180^0.

11. If n be integral $\tan(n\,180^0 + A) = \tan A^0$.

12. Find the ratios of $A - 90^0$ and $A - 180^0$.

13. $\operatorname{vers}(180^0 - A) + \operatorname{vers}(360^0 - A) = 2$.

14. $\cos^2 A + \cos^2(90^0 + A) + \cos^2(180^0 + A) + \cos^2(270^0 + A) = 2$.

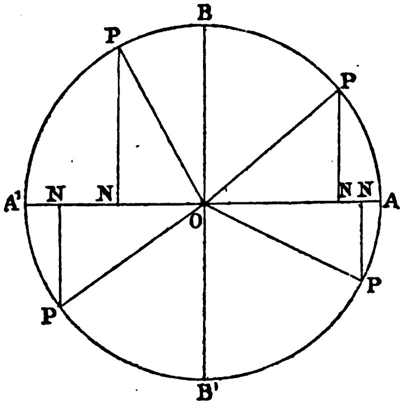
15. $\cot(-a)\operatorname{cosec}(-a)(1 - \cos^2 a) = \cos(-a)$.

CHAPTER VII.

CHANGES OF RATIOS AS THE ANGLE CHANGES.

N.B. This Chapter may be omitted the first time the book is read.

24. *To trace the change in sign and value of sin* A *as* A *changes from* 0° *to* 360°.



Describe a circle with OA as radius, O as centre, and draw two diameters AOA', BOB' at right angles.

Let the *generating line* OP revolve from OA through 360°; draw PN perpendicular to OA.

Let $OA = r$; then OP is *always positive* and is always equal to r.

Now,

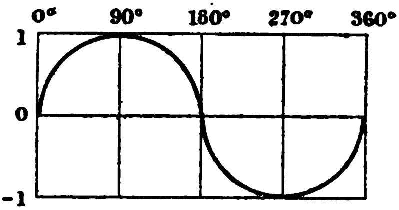
i. as OP revolves from OA to OB,
the angle AOP changes from 0 to 90°;
$\therefore$ PN changes from 0 to OB or r and is *positive;*
$\sin A$ or $\dfrac{PN}{OP}$ changes from 0 to $\dfrac{OB}{OB}$ or from 0 to 1 and is *positive.*

ii. As OP revolves from OB to OA',
the angle AOP changes from 90° to 180°;
PN changes from OB to 0 and is *positive;*
$\sin A$ or $\dfrac{PN}{OP}$ changes from 1 to 0 and is *positive.*

iii. As OP revolves from OA' to OB',
the angle AOP changes from 180° to 270°;
PN changes from 0 to OB' or $-r$, and is *negative;*
$\sin A$ or $\frac{PN}{OP}$ changes from 0 to -1 and is *negative.*

iv. As OP revolves from OB' to OA,
the angle AOP changes from 270° to 360°;
PN changes from OB' or $-r$ to 0, and is *negative;*
$\sin A$ or $\frac{PN}{OP}$ changes from -1 to 0, and is *negative.*

The annexed diagram shews these changes.



The curved line shews the variation in these values, the perpendicular distance of a point on it from the horizontal line through 0 representing the value of the sine for the corresponding angle.

25. *To trace the changes in the cosine.*

With the same figure,

as A changes from 0 to 90°,
ON .... r to 0 and is *positive;*
$\therefore \cos A$ or $\frac{ON}{OP}$ .... 1 to 0 and is *positive.*

As A changes from 90° to 180°,
ON changes from 0 to OA' or $-r$ and is *negative;*
$\therefore \cos A$ or $\frac{ON}{OP}$ .... 0 to -1 and is *negative.*

As A changes from 180° to 270°,

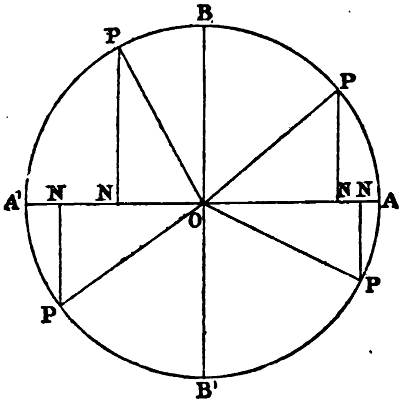
ON changes from $-r$ to 0 and is *negative;*

$\therefore$ $\cos A$ changes from -1 to 0 and is *negative.*

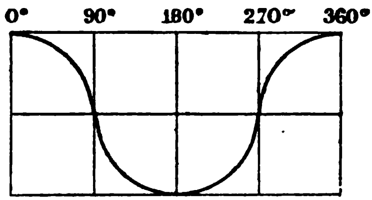
As A changes from 270° to 360°,

ON changes from 0 to r and is *positive;*

$\therefore$ $\cos A$ changes from 0 to 1 and is *positive.*



The annexed diagram shews these changes.



26. *To trace the changes in the tangent.*

With the same figure, in the first quadrant, from 0° to 90°,

PN changes from 0 to r, and ON from r to 0;

$$\therefore \tan A \text{ or } \frac{PN}{ON} \text{ changes from 0 to } \infty \text{ and is } \textit{positive.}$$

In the second quadrant, from 90° to 180°,

PN changes from r to 0 and ON from 0 to $-r$;

$$\therefore \tan A \text{ or } \frac{PN}{ON} \text{ changes from } -\infty \text{ to 0 and is } \textit{negative.}$$

In the third quadrant, from 180° to 270°,

PN changes from 0 to $-r$ and ON from $-r$ to 0,
both negative;

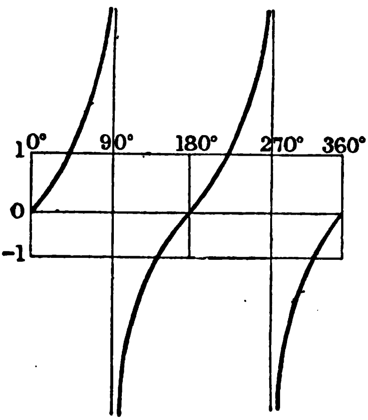
$\therefore$ tan A or $\frac{PN}{ON}$ changes from 0 to ∞ and is *positive.*

In the fourth quadrant, from 270° to 360°,

PN changes from $-r$ to 0, ON from 0 to r;

$\therefore$ tan A or $\frac{PN}{ON}$ changes from $-\infty$ to 0 and is *negative.*

The annexed diagram shews these changes.



The other ratios may be traced in the same way, or treated as reciprocals.

All the ratios change sign in passing through 0 or ∞.

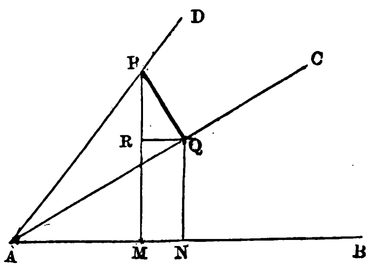
The first occurs when the numerator passes from a small positive to a small negative value, the second when the denominator does the same.

Examples on this chapter will be found among the miscellaneous questions at the end of the book.

CHAPTER VIII.

RATIOS OF COMPOUND ANGLES.

27. *To find the sine and cosine of* A + B *in terms of those of* A *and* B.



Let $BAC = A$, $CAD = B$;

$\therefore\ BAD = A + B$.

It is required to find the sine and cosine of $A + B$.

(The sine of $A + B$ is written $\sin(A + B)$, which, it must be observed, is not the same as $\sin A + \sin B$.)

Take any point P in AD, and draw PM, PQ perpendicular to AB, AC respectively.

Draw QN, QR perpendicular to AB, PM respectively.

Then, since PMA, PQA are right angles a circle will go round $APQM$;

$$\therefore\ QPM = QAM = A. \qquad \text{Euc. III. 21.}$$

Now

$$\sin(A + B) = \frac{PM}{AP} = \frac{QN + PR}{AP} = \frac{QN}{AQ} \cdot \frac{AQ}{AP} + \frac{PR}{PQ} \cdot \frac{PQ}{AP}.$$

$$\text{But } \frac{QN}{AQ} = \sin A, \quad \frac{AQ}{AP} = \cos B,$$

$$\frac{PR}{PQ} = \cos QPR = \cos A, \quad \frac{PQ}{AP} = \cos B;$$

$$\therefore\ \sin(A + B) = \sin A \cos B + \cos A \sin B$$

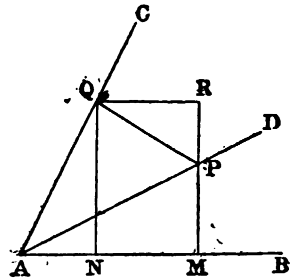
So $$\cos(A + B) = \frac{AM}{AP} = \frac{AN - QR}{AP} = \frac{AN}{AQ} \cdot \frac{AQ}{AP} - \frac{QR}{PQ} \cdot \frac{PQ}{AP}$$

$$= \cos A \cos B - \sin A \sin B.$$

28. *To find* $\sin(A-B)$ *and* $\cos(A-B)$.

Let $BAC=A$, $CAD=B$;

$\therefore\ BAD=A-B$.



Take any point P in AD, and draw PM, PQ perpendicular to AB, AC respectively.

Draw QN perpendicular to AB, and QR perpendicular to MP produced.

Then, as before, a circle goes round $AMPQ$;

$\therefore\ QAM+QPM=$ two right angles; Euc. III. 22.

$$\therefore\ QAN+QPM=RPQ+QPM;$$

$$\therefore\ RPQ=A.$$

$$\text{Now }\sin(A-B)=\frac{PM}{AP}=\frac{QN-PR}{AP}=\frac{QN}{AQ}\cdot\frac{AQ}{AP}-\frac{PR}{PQ}\cdot\frac{PQ}{AP}.$$

$$\text{But }\frac{QN}{AQ}=\sin A,\qquad \frac{AQ}{AP}=\cos B,$$

$$\frac{PR}{PQ}=\cos QPR=\cos A,\qquad \frac{PQ}{AP}=\sin B.$$

$$\therefore\ \sin(A-B)=\sin A\cos B-\cos A\sin B.$$

$$\text{So }\cos(A-B)=\frac{AM}{AP}=\frac{AN+QR}{AP}=\frac{AN}{AQ}\cdot\frac{AQ}{AP}+\frac{QR}{PQ}\cdot\frac{PQ}{AP}$$

$$=\cos A\cos B+\sin A\sin B.$$

Notice that the sign in the expression of the sine is the *same* as that in the angle expanded, in the cosine the *opposite*.

29. These formulæ can be proved for all values of A and B. It will be sufficient for the present to assume this and to shew how any one can be deduced from any other.

For instance, given $\sin(A+B)$, find $\cos(A-B)$.

$$\cos(A-B) = \sin(90^\circ - A + B) = \sin(\overline{90^\circ - A} + B)$$
$$= \sin(90^\circ - A)\cos B + \cos(90^\circ - A)\sin B$$
$$= \cos A \cos B + \sin A \sin B.$$

30. $$\operatorname{Tan}(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$
$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Divide both numerator and denominator by $\cos A \cos B$;

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

So $$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$$
$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B} = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

31. In the formulæ for $\sin(A+B)$, $\cos(A+B)$, $\tan(A+B)$ write B for A;

$$\therefore \sin 2A = 2\sin A \cos A,$$
$$\cos 2A = \cos^2 A - \sin^2 A,$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}.$$

We may write $1 - \sin^2 A$ for $\cos^2 A$ or $1 - \cos^2 A$ for $\sin^2 A$: thus we obtain

$$\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1.$$

The above are very important, the following less so.

$$\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$$
$$= 2\sin A \cos^2 A + (1 - 2\sin^2 A)\sin A$$
$$= 2\sin A(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A$$
$$= 3\sin A - 4\sin^3 A.$$

So $\cos 3A = 4\cos^3 A - 3\cos A$.

EXAMPLES ON CHAPTER VIII.

1. Find the sine, cosine, and tangent of 15^0, 75^0, 105^0, 165^0.

2. Find the sine, cosine, and tangent of 36^0 and 54^0.

3. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

4. $\sin 4A = 4\sin A\cos A\cos 2A$.

5. $\sin 8A = 8\sin A\cos A\cos 2A\cos 4A$.

6. If $\tan A = \frac{3}{4}$ and $\cos B = \frac{5}{13}$, find $\cos(A-B)$ and $\sin(A+B)$, A and B being acute angles.

7. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{40}{41}$, find $\sin(A+B)$ and $\tan(A-B)$ (i) when A is acute, (ii) when A is obtuse.

8. If $\tan A = \frac{1}{7}$ and $\tan B = \frac{1}{3}$, then the least value of $A+2B$ is 45^0.

9. If $\tan A = \dfrac{1}{\sqrt{3}}$ and $\tan B = \dfrac{1}{\sqrt{15}}$, then $\sin(A+B) = \sin 60^0 \cos 36^0$.

10. If $\tan A = \frac{1}{5}$ and $\tan B = \frac{1}{239}$, then $4A - B = 45^0$.

11. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$.

12. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

13. $\cos 4A = 8\cos^4 A - 8\cos^2 A + 1$.

14. $\dfrac{\cot A}{\cot 2A} = 1 + \sec 2A$.

15. $\tan A = \operatorname{cosec} 2A - \cot 2A$.

16. $\cot A = \operatorname{cosec} 2A + \cot 2A$.

17. $\dfrac{\sin A + \cos A}{\cos A - \sin A} = \tan 2A + \sec 2A$.

18. $\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C$.

19. $\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \cos B\sin C\sin A - \cos C\sin A\sin B$.

20. $\dfrac{\sin(A+B+C)}{\cos A\cos B\cos C} = \tan A + \tan B + \tan C - \tan A\tan B\tan C$.

21. $\dfrac{\cos(A+B+C)}{\sin A\sin B\sin C} = \cot A\cot B\cot C - \cot A - \cot B - \cot C$.

22. $\operatorname{vers}(180^0 - 2A) = 2\operatorname{vers}(90^0 + A)\operatorname{vers}(90^0 - A)$.

23. $\sec(x \pm y) = \dfrac{\sec x\sec y}{1 \pm \tan x\tan y}$.

24. $\tan A \pm \tan B = \sin(A \pm B)\sec A\sec B$.

25. $\dfrac{1 - 2\sin^2\theta}{1 + \sin 2\theta} = \dfrac{1 - \tan\theta}{1 + \tan\theta}$.

26. $\tan(45^0 + A) = \dfrac{1 + \tan A}{1 - \tan A} = \cot(45^0 - A)$.

27. $\tan(45^\circ - A) = \dfrac{1 - \tan A}{1 + \tan A} = \cot(45^\circ + A)$.

28. $\tan(45^\circ + A) - \tan(45^\circ - A) = 2\tan 2A$.

29. $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$.

30. $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$.

CHAPTER IX.

TRANSFORMATIONS. INVERSE RATIOS.

32. SINCE $\sin(A+B) = \sin A\cos B + \cos A \sin B$,

and $\sin(A-B) = \sin A \cos B - \cos A \sin B$;

$\therefore\ \sin(A+B) + \sin(A-B) = 2\sin A\cos B$,

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$.

Just in the same way, by addition and subtraction,

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$,

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$.

In these forms write C for $A+B$, D for $A-B$, and

$\therefore\ \dfrac{C+D}{2}$ for A, $\dfrac{C-D}{2}$ for B.

They become

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2};$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2};$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2};$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}.$$

The order of letters and signs in the fourth expression of each set should be noticed. These forms may be expressed verbally, and should be committed to memory. Thus: Twice the product of the sine of a greater angle and the cosine of a lesser is equal to the sum of the sines of the sum and difference.

Conversely: the sum of two sines is equal to twice the product of the sine of the semi-sum by the cosine of the semi-difference.

So for the others.

$$\sin A + \cos A = \sin A + \sin(90^\circ - A)$$
$$= 2\sin\frac{90^\circ - A + A}{2}\cos\frac{90^\circ - A - A}{2} = \sqrt{2}\cos(45^\circ - A).$$

So $\cos A - \sin A = \sqrt{2}\cos(45^\circ + A) = \sqrt{2}\sin(45^\circ - A)$.

33. No definite rules can be given for the use of these forms: in any transformation it is generally best to take one side, the more complicated one for choice, and try to reduce it.

Tangents and cotangents should generally be reduced to sines and cosines.

Generally, if it is required to prove that two expressions are equal, it is best to take one and work towards the other, *not* to alter both at once.

We give some examples.

i. $$\sin 4A + \sin 2A = 2\sin\frac{4A+2A}{2}\cos\frac{4A-2A}{2} = 2\sin 3A\cos A.$$

ii. $$\cos 3A - \cos 5A = 2\sin\frac{5A+3A}{2}\sin\frac{5A-3A}{2} = 2\sin 4A\sin A.$$

iii. $$4\cos A\cos B\cos C = 2\cos A\{\cos(B+C) + \cos(B-C)\}$$
$$= 2\cos A\cos(B+C) + 2\cos A\cos(B-C)$$
$$= \cos(A+B+C) + \cos(B+C-A) + \cos(A+C-B) + \cos(A+B-C).$$

iv. $$\sin(A+B-C) + \sin(A+C-B) + \sin(B+C-A) - \sin(A+B+C)$$
$$= 2\sin\frac{A+B-C+A+C-B}{2}\cos\frac{A+B-C-A-C+B}{2}$$
$$+ 2\cos\frac{B+C-A+A+B+C}{2}\sin\frac{B+C-A-A-B-C}{2}$$
$$= 2\sin A\cos(B-C) + 2\cos(B+C)\sin(-A)$$
$$= 2\sin A\{\cos(B-C) - \cos(B+C)\} = 4\sin A\sin B\sin C.$$

v. $$\cot A - \cot B = \frac{\cos A}{\sin A} - \frac{\cos B}{\sin B} = \frac{\sin B\cos A - \sin A\cos B}{\sin A\sin B} = \frac{\sin(B-A)}{\sin A\sin B}.$$

vi. $$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$$

$$=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos\frac{\alpha+\beta+2\gamma}{2}\cos\frac{\alpha+\beta}{2}$$

$$=2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta+2\gamma}{2}\right)$$

$$=4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta+2\gamma+\alpha-\beta}{4}\cos\frac{\alpha+\beta+2\gamma-\alpha+\beta}{4}$$

$$=4\cos\frac{\alpha+\beta}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\beta+\gamma}{2}.$$

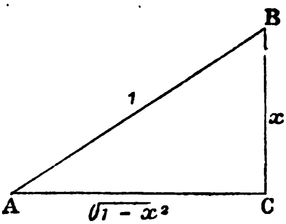
34. *Inverse functions.*

Suppose $\sin\alpha=x$, then α is the angle whose sine is x, and is written $\sin^{-1}x$.

So $\tan^{-1}x$ means the angle whose tangent is x.

Such expressions are called *inverse functions.*

Given $\sin\alpha=x$ *to find* $\tan\alpha$.



Form the right-angled triangle ABC, such that $BC=x$, $AB=1$; $\therefore$ $\sin BAC=x$. Then $BAC=\alpha$, $AC=\sqrt{1-x^2}$.

$$\therefore\ \tan\alpha=\frac{x}{\sqrt{1-x^2}},$$

or $$\alpha=\tan^{-1}\frac{x}{\sqrt{1-x^2}}.$$

To find $\tan^{-1}x+\tan^{-1}y$.

Let $\tan^{-1}x=\alpha$, $\tan^{-1}y=\beta$, $\therefore$ $\tan^{-1}x+\tan^{-1}y=\alpha+\beta$.

Now, (Art. 30), $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{x+y}{1-xy}$;

$$\therefore\ \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}.$$

Similarly, $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}.$

Notice that $\tan^{-1}1$ is 45°.

EXAMPLES ON CHAPTER IX.

Prove the following identities:

1. $2\sin 7A\cos 3A=\sin 10A+\sin 4A$.

2. $2\sin 3A\cos 5A=\sin 8A-\sin 2A$.

3. $\cos 5A\cos 6A=\frac{1}{2}(\cos 11A+\cos A)$.

4. $\sin 5A\sin 6A=\frac{1}{2}(\cos A-\cos 11A)$.

5. $\sin 3A+\sin A=2\sin 2A\cos A$.

6. $\sin 3A-\sin A=2\cos 2A\sin A$.

7. $\cos 5A+\cos A=2\cos 3A\cos 2A$.

8. $\cos A-\cos 5A=2\sin 3A\sin 2A$.

9. $\dfrac{\cos\alpha+\cos\beta}{\cos\alpha-\cos\beta}=\cot\dfrac{\alpha+\beta}{2}\cot\dfrac{\beta-\alpha}{2}$.

10. $\dfrac{\sin x-\sin y}{\cos y-\cos x}=\cot\dfrac{x+y}{2}$.

11. $\dfrac{\cos\theta+\cos 3\theta}{\cos 3\theta+\cos 5\theta}=\dfrac{1}{2\cos 2\theta-\sec 2\theta}$.

12. $\dfrac{\sin 3\theta+\sin\theta}{\sin 5\theta+\sin 3\theta}=\dfrac{\sec 2\theta}{2}$.

13. $\dfrac{\cos 3A+\cos 5A}{\cos 5A+\cos 7A}=\dfrac{\cos 4A}{\cos 6A}$.

14. $\dfrac{\cos A+\cos B}{\sin A-\sin B}+\dfrac{\sin A+\sin B}{\cos A-\cos B}=0$.

15. $\sin A+2\sin 3A+\sin 5A=4\sin 3A\cos^2 A$.

16. $\sin A+\sin B=\tan\dfrac{A+B}{2}(\cos A+\cos B)$.

17. $\tan 70^\circ+\tan 20^\circ=2\sec 50^\circ$.

18. $\cot A+\cot B=\sin(A+B)\operatorname{cosec} A\operatorname{cosec} B$.

19. $\dfrac{\sec A-\sec B}{\operatorname{cosec} A-\operatorname{cosec} B}+\tan\dfrac{A+B}{2}\tan A\tan B=0$.

20. $\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha=4\sin 4\alpha\cos 2\alpha\cos\alpha$.

21. $\cos\alpha-\cos 2\alpha+\cos 3\alpha-\cos 4\alpha=4\sin\dfrac{5\alpha}{2}\sin\dfrac{\alpha}{2}\cos\alpha$.

22. $\sin(60^\circ+A)=\sin(60^\circ-A)+\sqrt{3}\cos A$.

23. $\cot A-\cot 3A=\sin 2A\operatorname{cosec} A\operatorname{cosec} 3A$.

24. $\cos(A+B)\sin(A-B)+\cos(B+C)\sin(B-C)$
$+\cos(C+D)\sin(C-D)+\cos(D+A)\sin(D-A)=0$.

25. $\sin n\theta=2\sin\overline{n-1}\,\theta\cos\theta-\sin\overline{n-2}\,\theta$.

26. $\sin^2(A+B) = \sin^2 A + \sin^2 B + 2\sin A \sin B \cos(A+B)$.

27. $\cos^2(A-B) = \cos^2 A + \sin^2 B + 2\cos A \sin B \sin(A-B)$.

28. $\tan\theta \tan(60^0+\theta)\tan(60^0-\theta) = \tan 3\theta$.

29. If $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 = \frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi$, then

$$x = \frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\,a\,,\quad y = \frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\,b.$$

30. $\sin^{-1}x + \cos^{-1}x = 90^0$.

31. $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$.

32. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45^0$.

33. $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = 45^0 = \tan^{-1}\frac{5}{6} + \tan^{-1}\frac{1}{11}$.

34. $2\sin^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25} = \cos^{-1}\frac{7}{25}$.

35. $\sin^{-1}\frac{40}{41} = \cos^{-1}\frac{9}{41} = \tan^{-1}\frac{40}{9}$.

36. $\cot^{-1}\frac{4}{3} = \frac{1}{2}\cot^{-1}\frac{7}{24}$.

[In Questions 30—36 the angles are supposed to be acute.]

CHAPTER X.

RATIOS OF THE HALVES OF ANGLES.

35. *To find the ratios of* $\frac{A}{2}$ *in terms of* $\cos A$.

By Art. 31, $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.

In these forms write $\frac{A}{2}$ instead of A;

$$\therefore \cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}.$$

Hence $\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$, $\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$;

$$\therefore \cos\frac{A}{2} = \pm\sqrt{\left(\frac{1+\cos A}{2}\right)},\quad \sin\frac{A}{2} = \pm\sqrt{\left(\frac{1-\cos A}{2}\right)}.$$

If we know the value of A, we know the quadrant in which $\frac{A}{2}$ lies, and hence the signs of its ratios: there is therefore no ambiguity.

If, however, all we know about the angle is that it has a certain cosine, then if A be the acute or obtuse angle which has that cosine, the angle may be A or $360^\circ - A$ or any angle which differs from either of these by some multiple of 360°.

Hence the half angle may be $\frac{A}{2}$ or $180^\circ \pm \frac{A}{2}$ or $360^\circ - \frac{A}{2}$, or any angle differing from one of these by a multiple of 180°.

Now we have seen that the ratios of angles differing by multiples of 180° differ only in sign: hence the ambiguity of sign is explained.

$$\tan^2\frac{A}{2} = \frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}} = \frac{1-\cos A}{1+\cos A} = \frac{1-\cos^2 A}{(1+\cos A)^2} = \frac{\sin^2 A}{(1+\cos A)^2}:$$

$$\therefore \tan\frac{A}{2} = \frac{\sin A}{1+\cos A}.$$

There is no ambiguity of sign here, for $1+\cos A$ is positive, and $\sin A$ is of the same sign as $\tan\frac{A}{2}$, since they are both positive if $A < 180^\circ$, and $\sin A$ changes sign at every multiple of 180°, $\tan\frac{A}{2}$ at every multiple of 90°.

Since $\sin^2 A = 1 - \cos^2 A$,

$$\therefore \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}; \therefore \tan\frac{A}{2} = \frac{1-\cos A}{\sin A}.$$

36. *Given* $\sin A$ *to find* $\sin\frac{A}{2}$ *and* $\cos\frac{A}{2}$.

Since $\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1$, and $2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A$:

$$\therefore \left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2 = 1+\sin A;$$

$$\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2 = 1-\sin A:$$

$$\left.\begin{aligned}\therefore\ \cos\frac{A}{2}+\sin\frac{A}{2} &= \pm\sqrt{1+\sin A}\\ \cos\frac{A}{2}-\sin\frac{A}{2} &= \pm\sqrt{1-\sin A}\end{aligned}\right\}\ldots\ldots\ldots \text{(A)};$$

or, by addition and subtraction,

$$\left.\begin{aligned}2\cos\frac{A}{2} &= \pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}\\ 2\sin\frac{A}{2} &= \pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}\end{aligned}\right\}\ldots\ldots\text{(B)}.$$

37. In using these forms we must begin with (A) not (B), for since there are four ambiguities in (B) and only two easily determined ones in (A), the latter are preferable.

If we know the angle we can always tell the sign of its ratios and whether the sine or cosine is the greater; hence we can fix the sign in (A) for any special case.

It is better to use the forms of Art. 35 than those of Art. 36, since the ambiguities are more easily determined.

We will, by way of example, find the sines and cosines of 105° and 165° by both methods.

(i) **Since these angles are each $> 90^\circ < 180^\circ$, their sines are positive, and cosines negative.**

Hence

$$\sin 105^\circ = \sqrt{\frac{1-\cos 210^\circ}{2}} = \sqrt{\left(\frac{1+\frac{\sqrt{3}}{2}}{2}\right)} = \sqrt{\left(\frac{4+2\sqrt{3}}{8}\right)} = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

since $\cos 210^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$.

So $$\cos 105^0 = -\sqrt{\frac{1+\cos 210^0}{2}} = -\frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$\sin 165^0 = \sqrt{\frac{1-\cos 330^0}{2}} = \sqrt{\left(\frac{1-\frac{\sqrt{3}}{2}}{2}\right)} = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$\cos 165^0 = -\sqrt{\frac{1+\cos 330^0}{2}} = -\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

(ii) Since $105^0 > 45^0 < 135^0$, its sine > cosine *numerically;* $\therefore$ both $\sin 105^0 + \cos 105^0$ and $\sin 105^0 - \cos 105^0$ are positive.

$$\therefore \sin 105^0 + \cos 105^0 = +\sqrt{1+\sin 210^0} = \frac{1}{\sqrt{2}};$$

$$\sin 105^0 - \cos 105^0 = +\sqrt{1-\sin 210^0} = \frac{3}{2};$$

$$\therefore, \text{ as before, } \sin 105^0 = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

$$\cos 105^0 = -\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Again, since $165^0 > 135^0 < 225^0$, its cosine is numerically greater than its sine and negative (as may be seen by drawing a figure).

$$\therefore \cos 165^0 + \sin 165^0 = -\sqrt{1+\sin 330^0} = -\frac{1}{\sqrt{2}},$$

$$\cos 165^0 - \sin 165^0 = -\sqrt{1-\sin 330^0} = -\frac{\sqrt{3}}{\sqrt{2}}.$$

$$\therefore \sin 165^0 = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$\cos 165^0 = -\frac{\sqrt{3}+1}{2\sqrt{2}}, \text{ as before.}$$

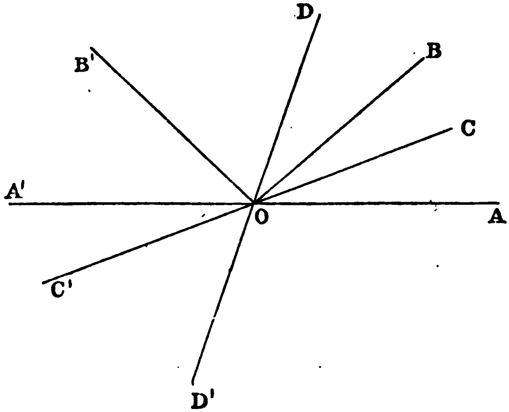
38. The reason why we obtain four values for the sine and cosine of $\frac{A}{2}$ may be explained geometrically thus.

Suppose that we know the sine of an angle and that it is positive.

Make the acute angle AOB having the given sine: produce AO to A' and make $A'OB' = AOB$. (See figure on p. 40.)

Then AOB' has the same sine as AOB, and so has every angle which differs from either of these angles by any multiple of 360°.

$\therefore$ A may be AOB, AOB' or any angle which differs from either by a multiple of $360°$.



Bisect AOB by OC and produce CO to C': bisect AOB' by OD and produce DO to D'.

Then $\frac{A}{2}$ may be either AOC, AOD, AOC', AOD' or any angle which differs from any of these by a multiple of $360°$.

AOC, AOC' have ratios differing only in *sign*, and so have AOD, AOD'.

But the ratios of AOC, AOD differ in magnitude.

Hence $\frac{A}{2}$ may be one of a set of four angles whose ratios differ in magnitude as well as sign.

A similar construction may be made if $\sin A$ be negative.

EXAMPLES ON CHAPTER X.

1. Find the ratios of $22° 30'$ and of $112° 30'$.

2. Find $\tan 7\frac{1}{2}°$ and $\cot 9°$.

3. Find forms for determining $\sin 3°$, $\sin 6°$, $\sin 21°$.

4. Shew that the sines of all multiples of $3°$ may be found by means of Arts. 32 and 35.

Prove that

5. $\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}$.

6. $\cos A=\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$.

7. $\tan^2\frac{A}{2}\tan A+2\tan\frac{A}{2}=\tan A$.

8. From this equation find $\tan\frac{A}{2}$ (i) when $A=60^0$, (ii) $A=300^0$.

9. $\tan^2(45^0+A)=\frac{1+\sin 2A}{1-\sin 2A}$.

10. $\frac{1+\sin\theta}{1+\cos\theta}=\frac{1}{2}\left(1+\tan\frac{\theta}{2}\right)^2$.

11. $2\cos 11^0\,15'=\sqrt{2+\sqrt{2+\sqrt{2}}}$.

In questions 12—17 $A<360^0$.

12. If $\sin A+\cos A=+\sqrt{1+\sin 2A}$, $A<135^0$ or $>315^0$.

13. If $\sin A+\cos A=-\sqrt{1+\sin 2A}$, A is between 135^0 and 315^0.

14. If $\cos A-\sin A=-\sqrt{1-\sin 2A}$, A is between 45^0 and 225^0.

15. Examine the case when $\cos A-\sin A=+\sqrt{1-\sin 2A}$.

16. If $2\cos A=-\sqrt{1+\sin 2A}+\sqrt{1-\sin 2A}$, $A>225^0$ and $<315^0$.

17. If $2\sin A=\sqrt{1+\sin 2A}+\sqrt{1-\sin 2A}$, $A>45^0$ and $<135^0$.

18. Affix the proper signs to the root symbols for $\sin\frac{A}{2}+\cos\frac{A}{2}$ and $\sin\frac{A}{2}-\cos\frac{A}{2}$ for the following values of A, 22^0, 122^0, 222^0, 322^0, 422^0, 522^0, 622^0.

19. Find $\cos 157\frac{1}{2}^0$ by 2 methods, shewing that your results agree.

20. Find $\tan 101\frac{1}{4}^0$ by Art. 35, finding the sine and cosine of $202\frac{1}{2}^0$ by Art. 36.

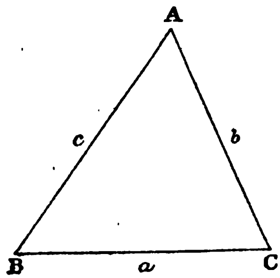
CHAPTER XI.

TRIANGLES.

39. A TRIANGLE has six parts or elements, its angles and sides. It is said to be solved when all are known.

Our object is to find equations connecting their values.

The sides opposite to A, B, C respectively are denoted by the letters a, b, c.



By Euc. i. 4, i. 8, i. 26, a triangle is determined if either two sides and the included angle, all the sides, or two angles and one side be known. Since then, if three quantities be given out of six, the others are determined, there must be three equations connecting these six quantities.

One of these equations is $A + B + C = 180°$....Euc. i. 32.

40. *The sides of a triangle are proportional to the sines of the angles opposite to them.*

Let ABC be a triangle and let B be one of the acute angles. (See the figures on the page opposite.)

Draw AD perpendicular to BC or BC produced.

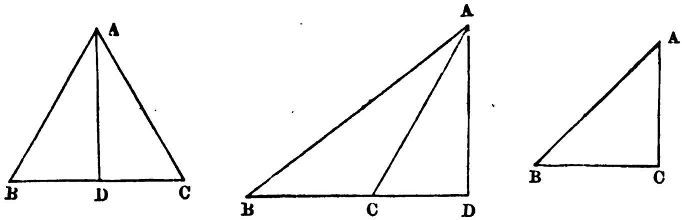
Then $\sin B = \dfrac{AD}{AB}$, $\therefore\ AD = AB \sin B = c \sin B$.

Also $\sin C = \sin ACD$, whether C be acute or obtuse.

$$\therefore\ \sin C = \frac{AD}{AC}, \text{ that is } AD = b \sin C.$$

$$\therefore\ b \sin C = c \sin B, \text{ or } \frac{b}{\sin B} = \frac{c}{\sin C}.$$

By interchanging the letters A and B, or by drawing the perpendicular from C, we can prove that each of these expressions is equal to $\frac{a}{\sin A}$.



If $C = 90^\circ$, D coincides with C, $\sin C = 1$, $\sin B = \frac{AC}{AB} = \frac{b}{c}$.

$$\therefore \text{ in all cases, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

For another proof of this important theorem, see Art. 46.

41. With the same figures and construction

$$BD = AB \cos B = c \cos B.$$

Now if C be acute,

$$CD = AC \cos C = b \cos C, \text{ and } BC = BD + DC,$$

or $a = c \cos B + b \cos C$.

If C be obtuse,

$$CD = AC \cos ACD = -AC \cos C = -b \cos C,$$

and $$BC = BD - DC \text{ or } a = c \cos B + b \cos C.$$

If C be a right angle $\cos C = 0$ and D coincides with C, therefore in all cases $a = c \cos B + b \cos C$.

Just in the same way we may prove that

$$b = c \cos A + a \cos C, \; c = a \cos B + b \cos A.$$

These forms are not independent of those in Art. 40, but may be proved from them thus;

$$\sin A = \sin (180^\circ - A) = \sin (B + C) = \sin B \cos C + \cos B \sin C.$$

$$\therefore \; a = \frac{a \sin B}{\sin A} \cos C + \frac{a \sin C}{\sin A} \cos B = b \cos C + c \cos B.$$

42. *To find the cosines of the angles.*

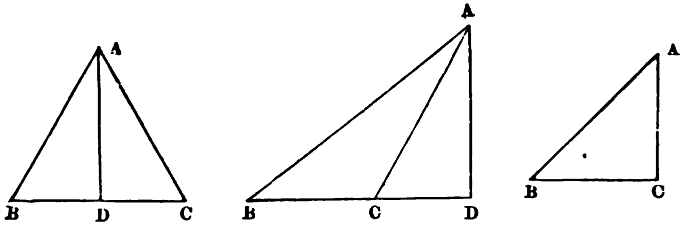
With the same figures and construction.

If C be acute

$$AB^2 = AC^2 + BC^2 - 2BC \cdot CD \ldots \text{Euc. ii. 13.}$$

And if C be obtuse

$$AB^2 = AC^2 + BC^2 + 2BC \cdot CD \ldots \text{Euc. ii. 12.}$$



Now $CD = \pm\, b \cos C$, as C is acute or obtuse, and vanishes if C be a right angle.

$$\therefore \text{ in all cases } c^2 = a^2 + b^2 - 2ab \cos C.$$

Hence $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Similarly

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

43. *To find the area of the triangle.*

With the same figures let S be the area.

Then by Euc. i. 41, $S = \frac{1}{2} BC \cdot AD$.

But $$AD = AB \sin B = c \sin B;$$

$$\therefore S = \tfrac{1}{2} ac \sin B.$$

By Art. 40, this expression may be written $\frac{1}{2} bc \sin A$, or $\frac{1}{2} ab \sin C$.

44. Since $A+B+C=180°$, $B+C=180°-A$,

$\therefore\ \sin A=\sin(B+C)$, $\cos A=-\cos(B+C)$,

$\tan A=-\tan(B+C)$.

In the same way, since

$$\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°,$$

$$\sin\frac{A}{2}=\cos\frac{B+C}{2},\ \cos\frac{A}{2}=\sin\frac{B+C}{2},$$

$$\tan\frac{A}{2}=\cot\frac{B+C}{2},\ \cot\frac{A}{2}=\tan\frac{B+C}{2}.$$

45. It is very inconvenient, for purposes of numerical calculation, to use results which consist of the sums and differences of squares, like the forms given for the cosines of the angles.

We will therefore obtain forms which involve factors only.

Let $$2s=a+b+c.$$

Subtract $2a$ from both sides,

$$\therefore\ 2(s-a)=b+c-a.$$

Similarly, $2(s-b)=a+c-b$, $2(s-c)=a+b-c$.

Now $$\cos C=\frac{a^2+b^2-c^2}{2ab};$$

$$\therefore\ 2\sin^2\frac{C}{2}=1-\cos C=\frac{2ab-a^2-b^2+c^2}{2ab}=\frac{c^2-(a-b)^2}{2ab}$$

$$=\frac{(c-a+b)(c+a-b)}{2ab}=\frac{2(s-a)(s-b)}{ab}.$$

$$\therefore\ \sin\frac{C}{2}=\left\{\frac{(s-a)(s-b)}{ab}\right\}^{\frac{1}{2}}.$$

Similarly, $\sin\dfrac{A}{2}=\left\{\dfrac{(s-b)(s-c)}{bc}\right\}^{\frac{1}{2}}$, $\sin\dfrac{B}{2}=\left\{\dfrac{(s-c)(s-a)}{ac}\right\}^{\frac{1}{2}}$.

Again

$$2\cos^2\frac{C}{2}=1+\cos C=\frac{(a+b)^2-c^2}{2ab}=\frac{2s(s-c)}{ab},$$

$$\therefore \cos\frac{C}{2}=\left\{\frac{s(s-c)}{ab}\right\}^{\frac{1}{2}},$$

and so we may write down the values of $\cos\frac{A}{2}$, $\cos\frac{B}{2}$.

From these

$$\tan\frac{C}{2}=\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}=\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}^{\frac{1}{2}}:$$

$$\sin^2 C=(1+\cos C)(1-\cos C)=\frac{4s(s-a)(s-b)(s-c)}{a^2b^2};$$

$$\therefore \sin C=\frac{2}{ab}\left\{s(s-a)(s-b)(s-c)\right\}^{\frac{1}{2}}:$$

$$S=\tfrac{1}{2}ab\sin C=\{s(s-a)(s-b)(s-c)\}^{\frac{1}{2}}.$$

Since any two sides of a triangle are greater than the third side, each factor under the root in any one of these expressions is positive, and all the expressions, being ratios of acute angles, are so too.

Hence there is no ambiguity of sign.

EXAMPLES ON CHAPTER XI.

ABC is a triangle; prove the following identities.

1. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\cot\dfrac{C}{2}$.

2. $\dfrac{\sin A-\sin B}{\cos B-\cos A}=\tan\dfrac{C}{2}$.

3. $\tan A+\tan B=\sin C\sec A\sec B$.

4. $\tan B-\cot A=\cos C\sec B\operatorname{cosec} A$.

5. $\tan A \tan B = 1 + \dfrac{\cos C}{\cos A \cos B}$.

6. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

7. $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$.

8. $\tan \dfrac{A}{2} \tan \dfrac{B}{2} + \tan \dfrac{B}{2} \tan \dfrac{C}{2} + \tan \dfrac{C}{2} \tan \dfrac{A}{2} = 1$.

9. $\sin A + \sin B + \sin C = 4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$.

10. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

11. $\dfrac{a+b}{a-b} = \dfrac{\sin A + \sin B}{\sin A - \sin B}$.

12. $\dfrac{a-b+c}{a+b+c} = \dfrac{\sin A - \sin B + \sin C}{\sin A + \sin B + \sin C}$.

13. $\cot A + \cot B = \dfrac{c}{a} \operatorname{cosec} B$.

14. $\dfrac{a-b+c}{a+b+c} = \tan \dfrac{A}{2} \tan \dfrac{C}{2}$.

15. $a \cos A - b \cos B = \dfrac{a}{2 \sin A} (\sin 2A - \sin 2B)$.

16. $a \cos A + b \cos B = c \cos (A - B)$.

17. $a + b + c = (a+b) \cos C + (a+c) \cos B + (b+c) \cos A$.

18. $2 \dfrac{a+b}{c} \sin^2 \dfrac{C}{2} = \cos A + \cos B$.

19. $c^2 \sin (A - B) = (a^2 - b^2) \sin C$.

20. $c^2 = (a+b)^2 \sin^2 \dfrac{C}{2} + (a-b)^2 \cos^2 \dfrac{C}{2}$.

21. $b^2 \cos^2 C - c^2 \cos^2 B = b^2 - c^2$.

22. $c = b \cos A + \sqrt{a^2 - b^2 \sin^2 A}$.

23. $\dfrac{\sin B + \sin (A - C)}{\sin A + \sin (B - C)} = \dfrac{c \cos B + b \cos C}{a \cos C + c \cos A}$.

24. $\dfrac{b}{c} \cos A + \dfrac{c}{a} \cos B + \dfrac{a}{b} \cos C = \frac{3}{2} + \frac{1}{2} \left(\dfrac{b}{c} - \dfrac{c}{b}\right) \left(\dfrac{c}{a} - \dfrac{a}{c}\right) \left(\dfrac{a}{b} - \dfrac{b}{a}\right)$.

25. $(a+b)^2 = c^2 + 4ab \cos^2 \dfrac{A+B}{2}$.

26. $S=\frac{1}{4}\{2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4\}^{\frac{1}{2}}=\dfrac{c^2\sin A\sin B}{2\sin C}$

$$=\tfrac{1}{2}(a^2-b^2)\frac{\sin A\sin B}{\sin(A-B)}=\frac{2s^2\sin A\sin B\sin C}{(\sin A+\sin B+\sin C)^2}.$$

27. $8S^2=abc\,(a\cos A+b\cos B+c\cos C)$.

28. If $(a^2+b^2)\sin(A-B)=(a^2-b^2)\sin(A+B)$, the triangle is either isosceles or right-angled.

29. If $(a^2+b^2)\cos 2A=b^2-a^2$, the triangle is right-angled.

30. D is the middle point of BC, shew that

$$AD^2=c^2+\frac{a^2}{4}-ac\cos B$$
$$=c^2+\frac{b^2}{4}-bc\cos A$$
$$=\frac{2(b^2+c^2)-a^2}{4}.$$

CHAPTER XII.

CIRCLES RELATED TO THE TRIANGLE.

46. *To find the radius of the circle described about the triangle.*

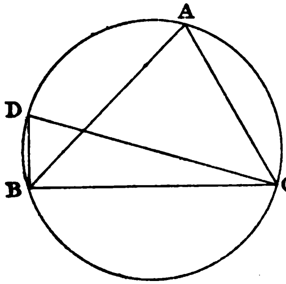
Let ABC be a triangle: describe a circle about it, and draw BD perpendicular to BC to meet the circle in D: join CD. (See the figures on the page opposite.)

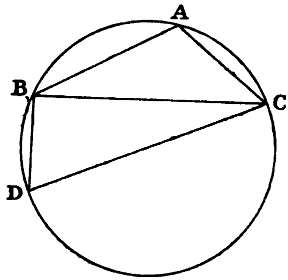
Let the radius of this circle be R; then $CD=2R$, since $CBD=90^\circ$.

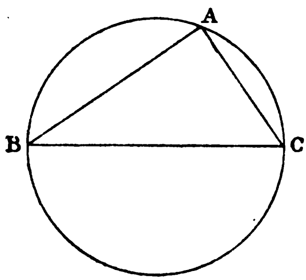
Now if A be acute, $BDC=BAC$; and if A be obtuse $BDC=180^\circ-BAC$. Therefore in both cases

$$\sin BDC=\sin A.$$

But $$\sin BDC = \frac{BC}{CD} = \frac{a}{2R}.$$







If $A = 90°$, $\sin BAC = 1$ and $a = 2R$; therefore in all cases

$$\sin A = \frac{a}{2R} \text{ or } \frac{a}{\sin A} = 2R.$$

Similarly

$$2R = \frac{b}{\sin B} \text{ and } 2R = \frac{c}{\sin C}.$$

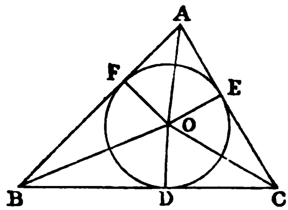
This gives us another proof that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The value of R may be written in the form

$$\frac{abc}{2bc \sin A} \text{ or } \frac{abc}{4S}.$$

47. *To find the radius of the circle inscribed in the triangle.*



Let O be the centre of the inscribed circle, join OA, OB, OC.

Draw OD, OE, OF perpendicular to the sides.

Let the radius OD be r.

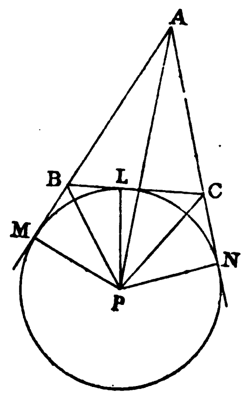
Then $\Delta BOC + \Delta COA + \Delta AOB = \Delta ABC$.

But $BOC = \frac{1}{2} BC \,.\, OD = \frac{1}{2}ar$; so $COA = \frac{1}{2}br$, $AOB = \frac{1}{2}cr$.

$$\therefore\ \tfrac{1}{2}(a+b+c)\,r = S,\ sr = S,\ r = \frac{S}{s}.$$

48. *To find the radii of the circles which touch one side of the triangle and the other two produced.*

Let P be the centre, r_1 the radius of the circle MLN, which touches BC in L, and AB, AC produced in M, N respectively. Join AP, BP, CP.



Then

$$\Delta ABC = \Delta APB + \Delta APC - \Delta BPC$$

or

$$S = \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1 - \tfrac{1}{2}ar_1 = (s-a)r_1,$$

$$\therefore\ r_1 = \frac{S}{s-a}.$$

Similarly, if r_2, r_3 be the radii of the circles which touch the sides opposite to B, C respectively, and the others produced,

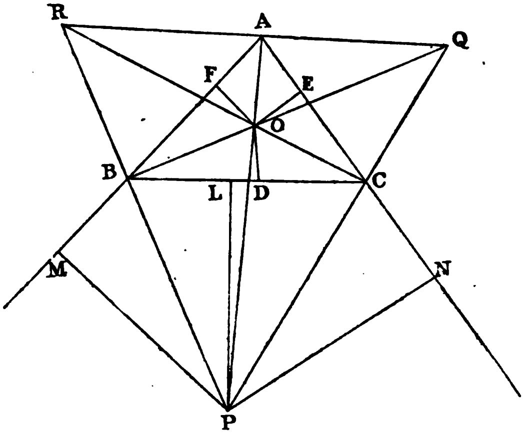
$$r_2 = \frac{S}{s-b},\ r_3 = \frac{S}{s-c}.$$

These are called the escribed circles.

49. Some properties of these circles are worth attention.

Let O be the centre of the inscribed circle, OD, OE, OF the perpendiculars from O on the sides.

Let P, Q, R be the centres of the escribed circles, P being opposite to A, Q to B, R to C.



Let PL, PM, PN be perpendicular to the sides.

Then since $OF = OE$, and $AF = AE$ (being tangents to the same circle from A), and AO is common, therefore OA bisects the angle A.

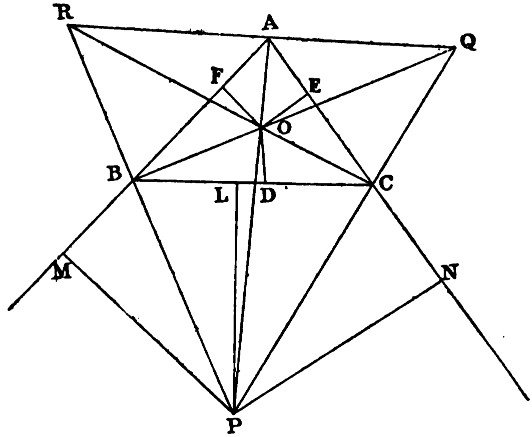
Similarly AP bisects the angle A, $\therefore AOP$ is a straight line.

So OB bisects the angle B, and, since $PL = PM$, $BL = BM$, and BP is common, therefore BP bisects CBM, that is $PBC = 90° - \frac{B}{2}$;

$$\therefore PBO = 90° - \frac{B}{2} + \frac{B}{2} = 90°,$$

similarly $RBO = 90°$, $\therefore RBP$ is a straight line.

So also BOQ, COR, PCQ, QAR are straight lines.



Again since the circle touches the sides in D, E, F; $AF = AE$, $BD = BF$, $CD = CE$.

If therefore we take these lines alternately beginning at any angular point, the alternate sets are equal and each equal to s.

Again $\quad BL = BM,\ CL = CN,\ AM = AN;$

$$\therefore\ AM + AN = \text{sum of sides};$$

$$\therefore\ AM = AN = s.$$

So, $\quad \because\ AF + CD + BF = s,$

that is $\quad AF + BC = s,\ AF = s - a,$

$$\therefore\ MF = AM - AF = a.$$

So $BM = s - AB = s - c = CD$.

Again $$OF = AF \tan\frac{A}{2},$$

$$\text{or } r = (s - a)\tan\frac{A}{2}.$$

In the same way $r_1 = s\tan\frac{A}{2}$, $OP = MF\sec\frac{A}{2} = a\sec\frac{A}{2}$.

EXAMPLES ON CHAPTER XII.

1. $R = \dfrac{s}{\sin A + \sin B + \sin C}$.

2. $4R \cos \dfrac{C}{2} = (a+b) \sec \dfrac{A-B}{2}$.

3. $4R \sin \dfrac{A}{2} = (b-c) \operatorname{cosec} \dfrac{B-C}{2}$.

4. $4R \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2} = s$.

5. If p be the perpendicular from A on BC, $2Rp = bc$.

6. $R^2 (\sin 2A + \sin 2B + \sin 2C) = bc \sin A$.

7. $r = 4R \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$.

8. $r_1 = 4R \sin \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$.

9. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$.

10. $r = s \tan \dfrac{A}{2} \tan \dfrac{B}{2} \tan \dfrac{C}{2}$.

11. $4sRr = abc$.

12. $r_1 + r_2 + r_3 - r = 4R$.

13. $rr_1 r_2 r_3 = s(s-a)(s-b)(s-c)$.

14. $r_1 r_2 = ab \cos^2 \dfrac{C}{2}$.

15. $r_1 r_2 + rr_3 = ab$.

16. If x, y, z be the perpendiculars from the centre of the circumscribing circle on the sides, $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = \dfrac{abc}{4xyz}$, and $x + y + z = R + r$.

17. If the straight line joining the centres of the inscribed and circumscribing circles pass through A, then $B = C$.

18. If the centres of the circumscribing and inscribed circles coincide, the triangle is equilateral.

19. O, P, Q, R are the centres of the inscribed and escribed circles, P being opposite to A, Q to B, R to C: shew that the angles of POQ are $\frac{B}{2}$, $\frac{A}{2}$, $90^0+\frac{C}{2}$, that those of PQR are $\frac{B+C}{2}$, $\frac{C+A}{2}$, $\frac{A+B}{2}$, that $OP=4R\sin\frac{A}{2}$, and that $PQ=4R\cos\frac{C}{2}$.

20. Shew that the circles circumscribing the triangles in question 19 are all equal, and that the radius of each is $2R$.

CHAPTER XIII.

LOGARITHMS.

50. *If* $a^x = m$, *then* x *is called the logarithm of* m *to base* a, *and is written* $\log_a m$.

The base a is always 10 in practice.

The logarithm of a number to a given base is therefore the index of the power to which the base must be raised in order to obtain the number.

51. The utility of logarithms depends on the following propositions:

(i) $\log_a (mn) = \log_a m + \log_a n$.

For let $a^x = m$, $a^y = n$; $\therefore a^{x+y} = mn$,

or $$x + y = \log_a (mn),$$

but $x = \log_a m$, $y = \log_a n$, $\therefore \log_a (mn) = \log_a m + \log_a n$.

(ii) $\log_a \frac{m}{n} = \log_a m - \log_a n$.

For if $a^x = m$, $a^y = n$, $a^{x-y} = \frac{a^x}{a^y} = \frac{m}{n}$, that is $x - y = \log_a \frac{m}{n}$.

(iii) $\log_a m^n = n \log_a m$.

For if $a^x = m$, $a^{nx} = m^n$, that is $nx = \log_a m^n$.

(iv) $\log_a \sqrt[n]{m} = \frac{1}{n} \log_a m$.

For if $a^x = m$, $a^{\frac{x}{n}} = m^{\frac{1}{n}}$, that is $\frac{x}{n} = \log_a \sqrt[n]{m}$.

Logarithms are calculated for all numbers of 7 figures or less. If then we want to find the product or quotient of two numbers, we have only to take the sum or difference of their logarithms, and look out the number of which the result is the logarithm.

Thus multiplication and division are reduced to addition and subtraction.

Again we can find the result of raising a number to any assigned power, or of extracting any root, by simply multiplying the logarithm by the exponent of the power, or dividing it by the number of the root.

52. Logarithms cannot be directly calculated to base 10, but to base e, where

$$e = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \text{\&c.}$$

ad infinitum; we therefore require a form for transposing a logarithm from one base to another.

Let $a^x = b^y = m$, so that $x = \log_a m$, $y = \log_b m$.

Then $a^{\frac{x}{y}} = b$, $\therefore \frac{x}{y} = \log_a b$ or $\log_a m = \log_a b \times \log_b m$.

So $\log_b m = \frac{1}{\log_a b} \log_a m$.

Hence, by writing e for a, and 10 for b, we obtain

$$\log_{10} m = \frac{1}{\log_e 10} \log_e m.$$

Thus if the logarithms of numbers be calculated to base e, and multiplied by $\frac{1}{\log_e 10}$, we shall get them to base 10.

$\frac{1}{\log_e 10}$ is called the modulus, and is equal to $\cdot 4342945$ nearly.

53. The value of any number, integral or fractional, obviously depends on two elements, the way in which its figures are arranged and the position of the decimal point. These two elements affect the logarithm in different ways. Any number, integral or fractional, positive or negative, may be written in the form $c+\mu$, where c is integral, but not necessarily positive, μ a positive proper fraction. Either c or μ may be zero.

If the logarithm of any number be written in the form $c+\mu$, c is called the characteristic, μ the mantissa.

Let N be any number, and let $\log_{10} N = c + \mu$;

$$\therefore \log_{10}(N \times 10^p) = \log_{10} N + \log_{10} 10^p = p + c + \mu,$$
$$\log_{10}(N \div 10^p) = \log_{10} N - \log_{10} p = c - p + \mu.$$

In each case we see that only the characteristic of the logarithm is affected by multiplying or dividing the number by any power of 10, that is by changing the position of the decimal point.

54. The characteristic of the logarithm of a number can be at once determined by inspection, and is one less than the number of figures before the decimal point.

For let N be a number of p figures, and let $10^x = N$. Now the least number of p figures is 10^{p-1} and the greatest that formed by p nines or $10^p - 1$;

$$\therefore 10^x \text{ cannot be } < 10^{p-1} \text{ or } > 10^p - 1,$$

$\therefore x = p - 1 + \mu$ where μ is a proper fraction; $\therefore p-1$ is the characteristic of $\log N$.

This rule can be extended to the case of proper fractions.

Let F be a fraction expressed as a decimal, and let q be the number of cyphers after the decimal point before the significant figures.

$\therefore$ $F \times 10^{q+1}$ is a number between 1 and 10, 10^{μ} say, where μ is a proper fraction;

$$\therefore \log (F \times 10^{q+1}) = \mu;$$
$$\log F = \mu - (q+1);$$

$-(q+1)$ is $\therefore$ the characteristic of F.

The mantissa is unaffected by the position of the decimal point.

For $\log N \times 10^p = p + \log N$.

In the tables the mantissæ only are given, the decimal point is omitted and the logarithms are calculated to 7 places for 5 significant figures.

N.B. log 1 is 0 to any base, for $a^0 = 1$.

Examples. Given $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$, find $\log 5$, $\log 864$, $\log \cdot 864$, $\log \cdot 003125$, $\log \sqrt[3]{\cdot 03}$, $\log \frac{64}{125}$.

$$5 = \tfrac{10}{2}; \therefore \log 5 = \log 10 - \log 2 = 1 - \cdot 30103 = \cdot 69897.$$
$$864 = 8 \times 108 = 32 \times 27 = 2^5 3^3;$$
$$\therefore \log 864 = \log 2^5 + \log 3^3 = 5 \log 2 + 3 \log 3 = 1 \cdot 50515 + 1 \cdot 4313639 = 2 \cdot 9365139,$$
$$\log \cdot 864 = \log \tfrac{864}{1000} = \log 864 - \log 1000 = \cdot 9365139 - 1.$$

This is always written $\bar{1} \cdot 9365139$, to shew that the characteristic is negative and the mantissa positive.

$$\text{Log} \cdot 003125 = \log \frac{5^5}{10^6} = 5 \log 5 - 6 = \bar{3} \cdot 49485.$$

Or thus, $\cdot 003125 = \frac{1}{320}$.

$$\text{Log} \left(\tfrac{1}{320}\right) = -\log 320 = -1 - 5 \log 2 = -2 \cdot 50515$$
$$= -3 + (1 - \cdot 50515) = \bar{3} \cdot 49485.$$

$$\text{Log} \sqrt[3]{\cdot 03} = \tfrac{1}{3} \log \cdot 03 = \frac{\bar{2} \cdot 4771213}{3},$$
$$= \frac{\bar{3} + 1 \cdot 4771213}{3} = \bar{1} \cdot 4923728.$$

Here we have to divide $-2 + \cdot 4771213$ by 3.

Make the negative integer divisible by 3, that is write $-3+1$ for -2, and we obtain the quotient in the proper form.

$$\text{Log } \tfrac{64}{729} = \log 64 - \log 729 = 6\log 2 - 6\log 3$$
$$= -6 \times \cdot 1760913 = -1\cdot 0565478 = \bar{2}\cdot 9434522.$$

Find the number of digits in 3^8, and the number of cyphers after the decimal point in $(\cdot 02)^6$.

$\text{Log } 3^8 = 8\log 3 = 3\cdot 8169704$, the characteristic of which is 3;

$\therefore$ 3^8 is a number of 4 figures since it lies between 10^3 and 10^4.

$$\text{Log } (\cdot 02)^6 = 6\log \cdot 02 = 6(\log 2 - 2) = 1\cdot 80618 - 12$$
$$= \overline{11}\cdot 80618, \text{ the number required is } \therefore 10.$$

EXAMPLES ON CHAPTER XIII.

1. Find $\log_2 4$, $\log_{12} 8$, $\log_8 2$, $\log_3 \sqrt{27}$, log 64 to base $\dfrac{1}{\sqrt{2}}$.

2. Find the characteristics of $\log_8 13$, $\log_{13} 3$, $\log_7 350$, $\log_7 333$, $\log_7 \frac{1}{2}$, $\log_{1000} 32872\cdot 93$.

3. How many positive integers are there whose logarithms to base 2 have 5 for a characteristic.

4. Given $\log_{10} 2 = 30103$, $\log_{10} 3 = \cdot 4771213$, $\log_{10} 7 = \cdot 845098$; find the logarithms to base 10 of the following ·

$$\sqrt[3]{6},\ \cdot 6,\ \cdot 0015,\ 4\cdot 05,\ 3\cdot 43,\ \sqrt[5]{72},\ \sqrt[3]{\tfrac{32}{105}},\ 1\cdot 5625,\ (\tfrac{1}{7})^{\frac{2}{3}},\ 6480.$$

5. Find the number of digits in 2^{17}.

6. Solve $8^x = 4^y$, $16^y = 32\,(2^x)$.

7. What is the least integral power to which $1\cdot 05$ must be raised, that it may exceed 2?

8. Given
$$\log 21 = 1\cdot 3222193,$$
$$\log 25 = 1\cdot 39794,$$
$$\log 28 = 1\cdot 447158;$$
find log 224, log 288, log 3, log 2, log 7, $\log_2 7$, $\log_7 30$, and $\log_{100} 27$.

9. Given $\log 6\cdot 4145 = \cdot 8071628$; find the numbers whose logarithms are $3\cdot 8071628$ and $\bar{3}\cdot 8071628$.

10. Given
$$\log 54 = 1\cdot 7323938,$$
$$\log 72 = 1\cdot 8573325;$$
find log 45.

CHAPTER XIV.

USE OF TABLES.

55. THE following theorem, for the proof of which the student is referred to more advanced books, must be assumed before the tables can be used.

If the difference of two numbers be small compared with either, the difference of their logarithms is proportional to that of the numbers.

We will give extracts from the tables with examples of their use.

No.	0	1	2	3	4	5	6	7	8	9	D
7063	8489892	9954	$\bar{0}015$	0077	0138	0199	0261	0322	0384	0445	61

In all tables the mantissæ only are given, the point is omitted. Here log 7063 has the mantissa ·8489892, so

$$\log 70631 = 4{\cdot}8489954,\ \log 70632 = 4{\cdot}8490015,\ \&c.$$

$D = 61$ means that the difference between each set of figures is 61 or strictly ·0000061 (in some instances it is 62, the real difference being some quantity between the two).

Required log 70·63674 :

$$\left.\begin{array}{l}\log 70{\cdot}63700 = 1{\cdot}8490322\\ \log 70{\cdot}63700 = 1{\cdot}8490261\end{array}\right\}\text{(from the tables)}$$
$$D \quad = \quad 61.$$

Let x be the difference for 74, $\therefore\ x = \frac{74}{100} \times 61 = 45$ to 2 places,

$$\therefore \log 70{\cdot}63674 = 1{\cdot}8490261 + {\cdot}0000045$$
$$= 1{\cdot}8490306.$$

Required the number whose log is 2·8489921.

The sequence of figures is intermediate between 70630 and 70631, for

$$\log 706{\cdot}31 = 2{\cdot}8489954$$
$$\log 706{\cdot}30 = 2{\cdot}8489892$$
$$D = 62$$
$$\log 706{\cdot}30 + x = 2{\cdot}8489921,$$
$$\therefore \text{diff. for } x = 29,$$
$$\therefore x \text{ (to 2 figures)} = \tfrac{29}{62} \times 100 = 46{\cdot}7, \text{ \&c.}$$

Now since 46·7 is nearer 47 than 46 we take $x = 47$,

$$\therefore \text{the number required is } 706{\cdot}3047.$$

56. Trigonometrical ratios.

Since the sine and cosine are always less than unity, their logarithms are negative.

For the convenience of work 10 is always added to the logarithms of the trigonometrical ratios, which are then called tabular logarithms and written L sin A, &c. Thus L tan A means $10 + \log \tan A$.

Extract from the tables

31°			
′	sine	cosine	′
17	9·7153937	9·9317679	43
18	9·7156015	9·9316911	42
′	cosine	sine	′
			58°

or L sin 31° 17′ = 9·7153937,
L cos 31° 17′ = 9·9317679,
L sin 58° 42′ = 9·9316911,
L cos 58° 42′ = 9·9156015.

Since $\cos(90° - A) = \sin A$, $\sin(90° - A) = \cos A$, if we form a table for sines and cosines of angles between 0 and 45°, the same read backwards will give cosines and sines of angles between 90° and 45°. So for the other ratios.

Ex. Find L sin 31° 17′ 24″.

$$\text{Diff. for } 60'' = {\cdot}0002078,$$
$$\therefore \text{diff. for } 24'' = \tfrac{2}{5} \text{ of } {\cdot}0002078 = {\cdot}0000831;$$
$$\therefore \text{L} \sin 31° 17' 24'' = 9{\cdot}7154768.$$

Find A when $L \cos A = 9 \cdot 9317102$.

Here $A = 31^{\circ}\, 17'\, x''$

$$\begin{array}{lll} L\cos 31^{\circ}\,17' = 9\cdot9317679 & \qquad & L\cos 31^{\circ}\,17' = 9\cdot9317679 \\ L\cos 31^{\circ}\,18'' = 9\cdot9316911 & & L\cos 31^{\circ}\,17'\,x'' = 9\cdot9317102 \\ \hline D \quad = \quad 768 & & \text{diff. for } x'' = \quad 577 \end{array}$$

$$\therefore\ x = \tfrac{577}{768} \text{ of } 60 = 45 \text{ nearly};$$

$$\therefore\ A = 31^{\circ}\, 17'\, 45''.$$

EXAMPLES ON CHAPTER XIV.

In these examples it may be assumed that

$$\log 2 = \cdot30103,\ \log 3 = \cdot4771213,\ \log 7 = \cdot845098.$$

1. Given $\log 13664 = 4\cdot1355779$, $\log 13665 = 4\cdot1356096$; find $\log 13\cdot66435$, and find the number whose logarithm is $2\cdot1356053$.

2. Given $\log 16826 = 4\cdot2259809$, $\log 16827 = 4\cdot2260067$; find $\log 168\cdot2637$, and the number whose logarithm is $\bar{3}\cdot2259947$.

3. Given $\log 1\cdot3287 = \cdot1234269$, $\log 13\cdot288 = 1\cdot1234596$; find $\log \cdot00132874$.

4. Multiply $32\cdot0576$ by $\cdot69665$, the following logarithms being known;

$$\begin{aligned} \log 3\cdot2057 &= \cdot5059229, \\ \log 3\cdot2058 &= \cdot5059364, \\ \log 6\cdot9665 &= \cdot8430146, \\ \log 2\cdot2332 &= \cdot3489276, \\ \log 2\cdot2333 &= \cdot3489471. \end{aligned}$$

5. Divide 320576 by $\cdot69665$, the logarithms in the preceding question being known, and in addition, $\log 4\cdot6016 = \cdot6629089$, $D = \cdot0000094$.

6. Find $(1\cdot05)^{10}$, given $\log 1\cdot6288 = \cdot2118678$, $D = 266$.

7. Find $(1\cdot25)^4$ to six places of decimals, given

$$\log 2\cdot4414 = \cdot3876389, \quad \log 2\cdot4415 = \cdot3876567.$$

8. Find $(6414529)^{\frac{1}{5}}$, given

$$\log 6\cdot4145 = \cdot8071628, \quad \log 2\cdot2984 = \cdot3614256,$$
$$\log 6\cdot4146 = \cdot8071696, \quad \log 2\cdot2985 = \cdot3614445.$$

9. Find the seventh root of 3115455, given

$$\log 31154 = 4\cdot4935138,$$
$$\log 31155 = 4\cdot4935278,$$
$$\log 8\cdot4653 = \cdot9276424,$$
$$\log 8\cdot4654 = \cdot9276475.$$

10. Find the seventh root of $\cdot00324$, given

$$\log 44092 = 4\cdot6443598, \quad \log 44093 = 4\cdot6443696.$$

11. Find $\sqrt[3]{10}$, given $\log 2\cdot1544 = \cdot3333263$, $D = 202$.

12. Find $\sqrt[5]{84}$, given $\log 2\cdot4258 = \cdot384855$, $D = \cdot000018$.

13. $\text{L} \tan A = 10 + \text{L} \sin A - \text{L} \cos A$.

14. $\text{L} \sec A = 20 - \text{L} \cos A$.

15. Given $\text{L} \sin 20^\circ\, 19' = 9\cdot5405903$,

$\text{L} \sin 20^\circ\, 20' = 9\cdot5409314$;

find $\text{L} \sin 20^\circ\, 19'\, 34''$, and $\text{L} \cos 69^\circ\, 40'\, 53''$.

16. Given $\text{L} \sin 38^\circ\, 48' = 9\cdot7969930$,

$\text{L} \sin 38^\circ\, 49' = 9\cdot7971501$;

calculate $\text{L} \sin 36^\circ\, 48'\, 37''$.

17. Given $\text{L} \cos 37^\circ\, 31' = 9\cdot8993697$,

$\text{L} \cos 37^\circ\, 32' = 9\cdot8992727$;

find $\text{L} \cos 37^\circ\, 31'\, 30''$.

18. Given $\text{L} \sin 32^\circ = 9\cdot7242097$,

$\text{L} \sin 32^\circ\, 1' = 9\cdot7244118$;

find the angle for which the L sin is $9\cdot7242198$.

19. Given $\text{L} \tan 31^\circ\, 13' = 9\cdot7824864$, diff. for $1' = 2849$; find

$$\text{L} \tan 31^\circ\, 13'\, 6'', \quad \text{L} \cot 58^\circ\, 46'\, 6'',$$

and the angle for which the tabular logarithm of the tangent is $9\cdot7823039$.

20. Find $\text{L} \sin 30^\circ$ and $\text{L} \cos 30^\circ$, and given

$$\text{L} \sin 30^\circ\, 1' = 9\cdot6991887, \quad \text{L} \cos 30^\circ\, 1' = 9\cdot9374577;$$

find the tabular logarithms of all the ratios of $30^\circ\, 0'\, 24''$.

21. Shew that $\text{L} \sin 2A = \log 2 + \text{L} \sin A + \text{L} \cos A - 10$, and from the results of the preceding question find $\text{L} \sin 60^\circ\, 2'$.

CHAPTER XV.

SOLUTION OF TRIANGLES.

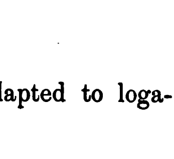
57. *A triangle is said to be solved when its sides and angles are known.*

If we know only the angles, we can only tell the relative magnitude of the sides: in order therefore to solve a triangle we must know at least one side.

We have the six parts of a triangle and three equations connecting them, it is therefore necessary to know three of these six parts, in order to determine the others.

58. *Right-angled triangles.*

Let C be a right angle.



We must therefore know two other parts of which one must be a side.

Our results must be adapted to logarithms.

There are four possible cases.

(i) a, b, being known to find A, B, c.

Here $c^2 = a^2 + b^2$; this however is not adapted to logarithms.

Also $\tan A = \dfrac{a}{b}$; $\therefore$ $\mathrm{L}\tan A = \log a - \log b + 10$.

This determines A; B is known since $B = 90^\circ - A$.

$c = b\sec A$ or $\log c = \log b + \mathrm{L}\sec A - 10 = \log b + 10 - \mathrm{L}\cos A$.

(ii) a, c known or b, c.

Here $b^2 = c^2 - a^2$, $\therefore 2 \log b = \log (c + a) + \log (c - a)$.

$$\sin A = \frac{a}{c}, \quad \therefore \text{L} \sin A = 10 + \log a - \log c.$$

(iii) a, A known, or a, B, since if one angle is known the other is.

Here $$c = a \operatorname{cosec} A,$$

$\therefore \log c = \log a + \text{L} \operatorname{cosec} A - 10 = \log a + 10 - \text{L} \sin A.$

$b = a \cot A$, $\therefore \log b = \log a + \text{L} \cot A - 10 = \log a + 10 - \text{L} \tan A$.

(iv) c, A known.

Here $a = c \sin A$, $b = c \cos A$, $B = 90^\circ - A$;

$\therefore \log a = \log c + \text{L} \sin A - 10$, $\log b = \log c + \text{L} \cos A - 10$.

59. *Oblique-angled triangles.*

There are four possible cases.

(i) When two angles and one side are given, as A, B, a.

(ii) When two sides and the included angle are given; a, b, C.

(iii) When two sides and an angle opposite to one of them are given; a, b, A.

(iv) When all the sides are given.

In determining angles from a known value of the sine, an ambiguity may arise, as two supplementary angles have the same sine, but in determining from the cosine or tangent or from any ratio of the half angle, no ambiguity can arise.

60. *To solve a triangle, given A, B, a.*

Here $C = 180^\circ - A - B$,

$$b = \frac{a \sin B}{\sin A}, \quad \therefore \log b = \log a + \text{L} \sin B - \text{L} \sin A;$$

$$c = \frac{a \sin C}{\sin A}, \quad \therefore \log c = \log a + \text{L} \sin C - \text{L} \sin A.$$

The triangle is $\therefore$ completely determined.

61. *To solve a triangle, given* a, b, C.

Here $c^2 = a^2 + b^2 - 2ab\cos C$, a form not adapted to logarithms; also $\sin B = \frac{b}{c}\sin C$.

The following solution is adapted to logarithms.

Since

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}};$$

$$\therefore \tan\frac{A-B}{2} = \frac{a-b}{a+b}\tan\frac{A+B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2};$$

$$\therefore \mathrm{L}\tan\frac{A-B}{2} = \log(a-b) - \log(a+b) + \mathrm{L}\cot\frac{C}{2}.$$

This gives $\frac{A-B}{2}$, and since $\frac{A+B}{2} = 90^\circ - \frac{C}{2}$, A and B are known; also $c = \frac{a\sin C}{\sin A}$, and $\therefore$ the triangle is completely determined.

The fraction $\frac{a-b}{a+b}$ should be reduced to its lowest terms before taking logarithms.

62. *Given* A, a, b, *to solve the triangle.*

Here $\sin B = \frac{b}{a}\sin A$, $\therefore \mathrm{L}\sin B = \log b - \log a + \mathrm{L}\sin A$.

Also $C = 180^\circ - (A+B)$;

$$c = \frac{a\sin C}{\sin A}; \ \therefore \log c = \log a + \mathrm{L}\sin C - \mathrm{L}\sin A.$$

Here, however, since there are two values of B which have a given sine, and these will give us two values of C

and therefore of c, there will be two solutions, unless we know that one is inadmissible. This is therefore called the *ambiguous case.*

We will examine it more closely.

We have always

$$\sin B = \frac{b}{a} \sin A.$$

Now if $a > b$, $A > B$ which is therefore acute, and there is no ambiguity.

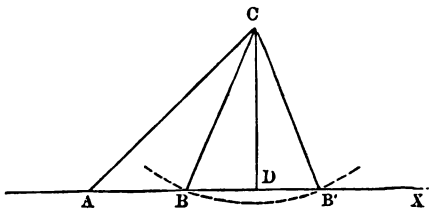
If $b \sin A = a$, $\sin B = 1$, $B = 90°$, and there is no ambiguity.

If $b \sin A > a$, $\sin B > 1$, and the triangle is impossible.

There is then an ambiguity *only* when a lies between b and $b \sin A$.

63. *To discuss the ambiguous case geometrically.*

Make CAX equal to the given value of A, and CA equal to the given value of b.



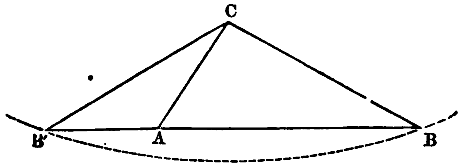
Then B must fall on AX.

Draw CD perpendicular to AX, and with centre C and radius equal to the given value of a describe a circle cutting AX in B, B'.

Then if B, B' are on the same side of A, both of the triangles BAC, $B'AC$ satisfy the given conditions.

In this case CB is less than CA, but greater than CD, or a lies between b and $b \sin A$.

If $CB > CA$, the circle will cut AX in two points on opposite sides of A; there will therefore be only one triangle: for since CAB' is not equal to the given value of A, the triangle CAB' does not satisfy the given conditions.



If $CB = CD$, the circle will touch AX, and the two triangles will coincide.

If $CB < CD$, the circle will not meet the line, and the triangle will be impossible.

64. *The sides being given, to solve the triangle.*

Here, since we know a, b, c, we know s, $s-a$, &c., therefore we may use any of the formulæ

$$\sin\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}},$$

$$\cos\frac{A}{2} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}},$$

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s.(s-a)}\right\}}.$$

The last is generally most convenient.

Taking logarithms we get

$$\text{L}\tan\frac{A}{2} - 10 = \log(s-b) + \log(s-c) - \log s - \log(s-a).$$

Having found one angle A, we can use the corresponding forms for B and C, or else

$$\sin B = \frac{b}{a}\sin A, \ \sin C = \frac{c}{a}\sin A,$$

where, if we take the greatest angle first, there can be no ambiguity.

EXAMPLES ON CHAPTER XV.

In the solution to the first 20 *questions, logarithms are not required.*

Solve the six following triangles, C being a right angle.

1. $a=90$, $b=90$.
2. $a=3$, $b=3\sqrt{3}$.
3. $a=7$, $c=14$.
4. $a=15$, $A=60^\circ$.
5. $c=12$, $A=30^\circ$.
6. $a=3$, $\sin A=\frac{3}{5}$.
7. If $C=90^\circ$, $\tan(A-B)=\dfrac{a^2-b^2}{2ab}$.

8. Standing opposite one corner of a house, 45 feet high, I find that its length subtends an angle whose tangent is 2, while its height subtends an angle whose tangent is $\frac{3}{5}$; find the length of the house and my distance from it.

9. AB is one mile, $ABC=45^\circ$, $BAC=75^\circ$; find AC in yards and the shortest distance from C to AB.

10. If $a=3$, $b=1$, $C=60^\circ$; find $\sin A$, $\sin B$ and c.

11. $b=3\sqrt{2a}$, $C=45^\circ$; find c and the sines of A and B.

12. $a=20$, $b=10$, $C=60^\circ$; solve the triangle.

13. If c_1, c_2 be the two values of c in the ambiguous case, then
$$c_1+c_2=2b\cos A,\quad c_1c_2=b^2-a^2.$$

14. If $a=1$, $b=2$, $A=30^\circ$, is the triangle ambiguous?

15. Given a, b and that $A-B=D$; solve the triangle.

16. Given S, s, A; solve the triangle.

17. The sides are 2, 3, 4; find the cosines of the angles, and R.

18. The sides are 3, 5, 7; find the cosine of the least angle, and the area.

19. The sides are 2, $\sqrt{2}$, and $\sqrt{3}-1$; find the angles.

20. The sides are 5, 8, 11; find the tangent of the least half angle, and the radius of the inscribed circle.

In the following questions, it may be assumed that $\log 2=\cdot 30103$, $\log 3=\cdot 4771213$, $\log 7=\cdot 845098$.

21. $a=3$, $b=7$, $C=90^\circ$; solve the triangle; given
$$\text{L}\tan 22^\circ 46' 12''=9\cdot 6320233,$$
$$\log 58=1\cdot 7634280,\quad \log 76158=4\cdot 8817174.$$

22. $c=85$, $a=43$, $C=90^0$; find b.

Given $\log 7 \cdot 3202 = \cdot 865229$.

23. $a=576 \cdot 12$, $c=873 \cdot 14$, $C=90^0$; find A.

Given $\log 57612 = 4 \cdot 7605054$, $\log 87314 = 4 \cdot 9410839$,

$L \sin 41^0 17' = 9 \cdot 8194012$, diff. for $1' = \cdot 0001438$.

24. $a=1000$, $A=33^0$, $C=90^0$; find c.

Given $L \sin 33^0 = 9 \cdot 7361088$,

$\log 18360 = 4 \cdot 2638727$,

$\log 18361 = 4 \cdot 2638963$.

25. $c=1000$, $A=44^0 25' 37''$, $C=90^0$ find a.

Given $L \sin 44^0 25' = 9 \cdot 845018$,

$L \sin 44^0 26' = 9 \cdot 845147$.

26. Find b; given $a=1000$, $A=50^0$, $B=64^0$,

$L \sin 50^0 = 9 \cdot 884254$,

$L \sin 64^0 = 9 \cdot 9536602$,

$\log 1 \cdot 17329 = \cdot 0694062$.

27. Find b, given $B=32^0 15'$, $C=21^0 47' 20''$. $a=34$.

Given also

$\log 3 \cdot 4 = \cdot 531479$, $\log 2 \cdot 241 = \cdot 350442$, $\log 2 \cdot 242 = \cdot 350636$,

$L \sin 32^0 15' = 9 \cdot 727228$, $L \sin 54^0 2' = 9 \cdot 908141$, diff. for $1' = \cdot 000092$.

28. Find b and c, given $B=48^0$, $C=54^0$, $a=38$.

Given also

$\log 38 = 1 \cdot 5797836$, $L \sin 54^0 = 9 \cdot 9079576$,

$\log 28 \cdot 865 = 1 \cdot 4603927$, $L \sin 48^0 = 9 \cdot 8710735$,

$\log 344251 = 1 \cdot 4972768$, $L \sin 78^0 = 9 \cdot 9904644$.

29. Find B and C, given $b=15$, $c=5$, $A=54^0$.

Given $L \cot 27^0 = 10 \cdot 2928341$,

$L \tan 44^0 27' = 9 \cdot 9916616$,

$L \tan 44^0 28' = 9 \cdot 9919143$.

30. Find A and B, given $a=135$, $b=105$, $C=60^0$,

$L \tan 12^0 12' = 9 \cdot 33448711$, $L \tan 12^0 13' = 9 \cdot 3354823$.

31. Find A and B, given $a=456 \cdot 12$, $b=296 \cdot 86$, $C=74^0 20'$.

Given also

$\log 15926 = 4 \cdot 2021067$, $\log 75298 = 4 \cdot 8767834$,

$L \cot 37^0 10' = 10 \cdot 1202593$, $L \tan 15^0 35' = 9 \cdot 4455352$,

$L \tan 15^0 36' = 9 \cdot 4459232$.

32. Find A, B, and $\log c$, given $a=12$, $b=13$, $C=60^0$,

$$\text{L}\tan 3^0\,57'=8\cdot8391633,\quad \text{L}\tan 3^0\,58'=8\cdot8409977,$$
$$\text{L}\cos 23^0\,57'=9\cdot9608987,\quad \text{L}\cos 23^0\,58'=9\cdot9607864.$$

33. Find B and C, given $A=50^0$, $b=119$, $a=97$.

Given also

$$\log 1\cdot19=\cdot075547,\qquad \log 9\cdot7=\cdot9867717,$$
$$\text{L}\sin 50^0=9\cdot884254,\quad \text{L}\sin 70^0=9\cdot9729858,$$
$$\text{L}\sin 70^0\,1'=9\cdot9730318.$$

34. Find A, B, b, given $a=25$, $c=24$, $C=65^0\,59'$.

Given also $\quad \text{L}\cos 24^0\,1'=9\cdot9606739,$

$$\log 1755=3\cdot2442771,\qquad \text{L}\sin 72^0\,4'=9\cdot9783702,$$
$$\log 1756=3\cdot2445245,\qquad \text{L}\sin 72^0\,5'=9\cdot9784111.$$
$$\log 2\cdot7905=\cdot4456820,\quad \text{L}\sin 41^0\,56'\,12''=9\cdot8249771.$$
$$\log 2\cdot7906=\cdot4456976,\qquad \text{L}\sin 6^0\,5'\,48''=9\cdot0261504.$$

35. Given two triangles in each of which $A=30^0$, $c=250$, but in one $a=125$, in the other $a=200$; solve the triangles in both cases.

Given $\quad \text{L}\sin 38^0\,41'=9\cdot79588,\qquad \log 61654=4\cdot78096,$

$$\text{L}\sin\ 8^0\,41'=9\cdot17890,\quad \log 372\cdot502=2\cdot5712827.$$
$$\text{L}\sin 68^0\,41'=9\cdot9692227.$$

36. The sides of a triangle are 5, 12, 13; determine the angles.

Given $\quad \text{L}\tan 11^0\,18'\,30''=9\cdot300967,$

$$\text{L}\tan 11^0\,18'\,40''=9\cdot301076.$$

37. Given $a=275\cdot35$, $b=189\cdot28$, $c=301\cdot47$; find A.

Given $\quad \log 38305=4\cdot5832555,\qquad \log 8158=3\cdot9115837,$

$$\log 10770=4\cdot0322157,\quad \text{L}\tan 31^0\,45'=9\cdot7915635,$$
$$\log 19377=4\cdot2872865,\quad \text{L}\tan 31^0\,46'=9\cdot7918458.$$

38. Find the greatest angle in a triangle whose sides are 7, 8, 9.

Given $\quad \text{L}\cos 36^0\,42'=9\cdot9040529$, diff. for $60''=\cdot0000942$.

39. Find the least angle in a triangle whose sides are 8, 10, 12.

Given $\quad \text{L}\sin 20^0\,42'=9\cdot5483585,$

$$\text{L}\sin 20^0\,43'=9\cdot5486927.$$

40. $a=4439$, $b=4861$, $c=8583$, find B.

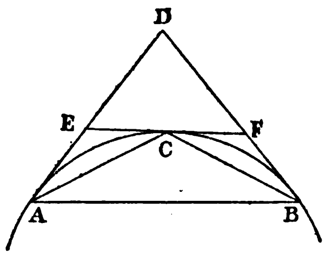
Given $\quad \log 8\cdot9415=\cdot9514104,\qquad \log 8\cdot583=\ \cdot9336391,$

$$\log 4\cdot439\ =\cdot6472851,\quad \text{L}\cos 11^0\,52'=9\cdot9906180,$$
$$\log 4\cdot0805=\cdot6107134,\quad \text{L}\cos 11^0\,53'=9\cdot9905914.$$

CHAPTER XVI.

CIRCULAR MEASURE.

65. If AB be a straight line, ACB any arc, AD, DB tangents at its ends, then clearly the arc $ACB > AB$. Take any point C in the arc; join AC, CB and draw the tangent ECF, then



$$AC + CB > AB.$$

Also since $DE + DF > EF$, by adding $AE + FB$ to each we find that

$$AD + DB > AE + EF + FB.$$

If we continue this process, and take other points in the arc and draw tangents at them, we form two polygons, one inside the arc, one outside, and the more points we take the more we increase the perimeter of the inscribed polygon, and diminish that of the outer.

But the arc lies between them.

Hence we assume:

(i) The arc of a circle is greater than the chord joining its ends, and less than the sum of the tangents at its ends.

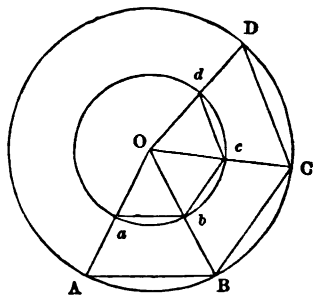
(ii) If a polygon be inscribed in a circle, and another described about it, the circumference of the circle is greater than the perimeter of the first, and less than that of the second.

(iii) If the number of sides of these polygons be indefinitely increased they become equal in perimeter and area, and so too are equal in both respects to the circumference and area of the curve between them.

These statements may be assumed without further proof.

66. *The circumference of any circle is proportional to its radius.*

Let O be the centre of two concentric circles ABC, &c., abc, &c.



Draw radii OaA, ObB, &c., and join AB, BC, ab, bc, &c.

Then Oab, OAB are similar triangles, so are

$$Obc, \quad OBC,$$

$$Ocd, \quad OCD, \text{ \&c.}$$

$$\therefore AB : ab :: AO : aO,$$

$$BC : bc :: AO : aO,$$

$$\text{\&c.} \qquad \text{\&c.}$$

$$\therefore AB + BC + CD + \dots : ab + bc + cd + \dots :: AO : aO.$$

Therefore perimeter of first polygon : perimeter of second :: radius of first circle : radius of second.

This proportion subsists if the number of sides of each polygon be indefinitely increased.

But then they ultimately coincide with the circumferences of the circles.

Therefore the circumferences of circles vary as their radii. The proof would have been the same if the polygons had been described about the circles.

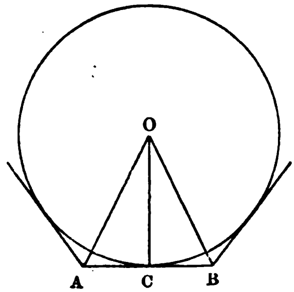
The ratio $\dfrac{\text{circumference}}{\text{diameter}}$ is called π: it cannot be exactly calculated: its value to five places of decimals is 3·14159.

$\frac{22}{7}$ is an approximation; $\frac{355}{113}$ is nearer the real value.

The former is generally taken in calculations which do not require great accuracy.

67. *If* r *be the radius of a circle, its circumference is* $2\pi r$, *and its area* πr^2.

For the ratio of the circumference to the diameter is π, therefore the circumference $= 2\pi r$.



Again, let O be the centre, AB a side of a regular polygon of n sides described about the circle, OC perpendicular to AB.

Let A be the area of the polygon, P the perimeter.

Now

$$\triangle OAB = \tfrac{1}{2} AB.OC = \tfrac{1}{2} AB.r.$$

Also

$$A = n\triangle OAB, \quad P = nAB.$$

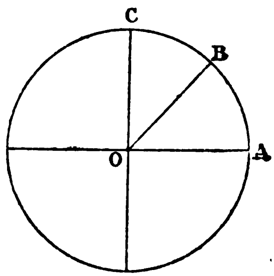
$\therefore A = \frac{1}{2}Pr$, for all values of n.

But when the number of sides is indefinitely increased the perimeter and area of the polygon coincide with those of the circle; that is when $P = 2\pi r$, $A = \pi r^2$.

The area of a circle of radius r is therefore πr^2.

68. *The angle subtended at the centre of a circle by an arc equal in length to the radius is constant.*

Let the arc AB be equal to the radius OA. Let OC be perpendicular to OA.



Now

$$AC = \tfrac{1}{4}\text{ circumference} = \frac{\pi}{2} r.$$

But by Euc. VI. 33,

$$\angle AOB : \angle AOC :: AB : AC,$$
$$:: r : \frac{\pi r}{2},$$
$$:: 2 : \pi,$$
$$\therefore AOB = \frac{2}{\pi} AOC.$$

AOB is $\therefore$ a fixed part of a right angle, i.e. it is a fixed angle.

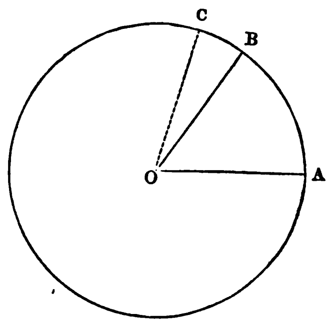
Since this angle is fixed it may be taken as the unit; angles measured by it are said to be expressed in circular measure.

The value of the unit of circular measure in degrees is easily found.

For $AOB : 90^\circ :: 2 : \pi$,

$$AOB = \frac{2}{\pi} \times 90^\circ = \frac{280^\circ}{3 \cdot 14159} = 57^\circ \cdot 29577.$$

69. *The circular measure of any angle is equal to the fraction* $\frac{l}{r}$, *where* l *is the length of an arc which subtends the angle at the centre of a circle whose radius is* r.



For, let AOC be any angle.

With centre O, radius OA (r), describe a circle. Let $AB = OA$.

Then, by Euc. VI. 33,

$$\frac{AOC}{AOB} = \frac{AC}{AB} = \frac{l}{r},$$

or $$AOC = \frac{l}{r} \times AOB.$$

If then $AOB = 1$, $AOC = \frac{l}{r}$.

If an angle be simply expressed by a number or a Greek letter, it is understood to be in circular measure.

70. If D be the number of degrees in an angle, G of grades, θ of circular units,

$$\frac{D}{180} = \frac{G}{200} = \frac{\theta}{\pi}.$$

For each fraction represents the ratio of the angle to two right angles.

Hence, the circular measure of a right angle is $\frac{\pi}{2}$.

π is the circular measure of two right angles. This statement is sometimes written in the form $\pi = 180^\circ$, a statement which can lead to no inaccuracy, if we remember that π here means π *times the angle* 1, not the number π.

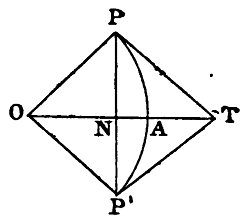
Two cautions must be given.

(1) The unit of circular measure *is not* π.

(2) The statement $\pi = 180$, the degree symbol being omitted, is *totally inaccurate* and misleading.

71. *If θ be the circular measure of an acute angle, $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude.*

For let AOP be an acute angle.



With centre O, any radius OA (r), describe an arc AP.

Draw PN perpendicular to OA, and produce it to meet the circle at P'; draw PT, $P'T$ touching the circle at PP'.

Then since PAP' lies between PP' and $PT + P'T$, AP lies between PN and PT,

$\therefore \frac{PN}{OP}, \frac{AP}{OP}, \frac{PT}{OP}$ are in ascending order of magnitude:

that is $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude.

72. When θ is very small $\frac{\sin\theta}{\theta}$, $\frac{\tan\theta}{\theta}$ are very nearly equal to unity.

For when $\theta = 0$, $\cos\theta = 1$.

Now $\sin\theta$, θ, $\tan\theta$ are in ascending order,

$$\therefore\ 1,\ \frac{\theta}{\sin\theta},\ \frac{1}{\cos\theta}\ \text{are so.}$$

But as θ becomes very small, $\frac{1}{\cos\theta}$ becomes very nearly equal to 1.

So $\therefore$ does $\frac{\theta}{\sin\theta}$, and $\therefore$ $\frac{\theta\cos\theta}{\sin\theta}$ or $\frac{\theta}{\tan\theta}$.

This is often stated shortly thus:

When $\qquad \theta = 0,\ \frac{\theta}{\sin\theta} = 1.$

Such a statement is only an abbreviation for the foregoing.

73. To find a limit of error in taking θ for $\sin\theta$.

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2}.$$

Now $\tan\frac{\theta}{2} > \frac{\theta}{2}$, and $\cos^2\frac{\theta}{2} = 1 - \sin^2\frac{\theta}{2}$; $\therefore$ $\cos^2\frac{\theta}{2} > 1 - \frac{\theta^2}{4}$.

Therefore $\sin\theta > \theta - \frac{\theta^3}{4}$.

But $\sin\theta < \theta$, $\therefore$ the error $< \frac{\theta^3}{4}$.

Thus if the angle be $1'$, $\theta = \frac{\pi}{180 \times 60} = \frac{22}{7 \times 10800}$ nearly, that is, $\theta < \cdot 0003$, $\frac{\theta^3}{4} < \cdot 000000000007$, or the error is 0 to eleven decimal places. The error can be proved to be less than $\frac{\theta^3}{6}$, but the proof is not suited to an elementary work.

EXAMPLES ON CHAPTER XVI.

1. Reduce to circular measure 15^0, 135^0, $36^0\ 3'\ 36''$, $125^g\ 31^{`}$, $25^{``}$, π^0, $\frac{\pi^g}{3}$.

2. Reduce to degrees $\frac{\pi}{5}$, $\frac{5}{3}\pi$, 1, $\frac{\pi+1}{4}$, $\cdot 314159$.

3. Express in circular measure an angle of a regular pentagon.

4. The circular measure of an angle of a regular polygon of n sides is $\frac{n-2}{n}\pi$.

5. The angles of a triangle are as $3:5:7$; express them in circular measure.

6. Find the number of degrees in the angle subtended at the centre of a circle whose radius is 1 foot by an arc of 5 inches.

7. ABC is a triangle, and the angles when expressed, one in degrees, one in grades, one in circular measure, are as $252 : 320 : \pi$; find them.

8. ABC is a triangle, and the number of degrees in A = number of grades in B = circular measure of C; find the angles in circular measure.

9. The sum of all the angles of a polygon is 10π: how many sides has it?

10. An exterior angle of a regular polygon is $\frac{1}{6}$ of an interior angle: express each in circular measure, and find the number of sides of the polygon.

11. The perimeter of a regular polygon of n sides, inscribed in a circle of radius r, is $2nr \sin\frac{\pi}{n}$.

12. The perimeter of a regular polygon of n sides, described about a circle of radius r, is $2nr \tan\frac{\pi}{n}$.

13. The radii of the circles inscribed in and described about a regular polygon of n sides, one of whose sides is a, are $\frac{a}{2}\cot\frac{\pi}{n}$, and $\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$.

14. The area of a regular polygon of n sides, one of whose sides is a, is $\frac{n}{2}a^2\cot\frac{\pi}{n}$.

15. The area of a regular polygon of n sides, inscribed in a circle of radius r, is $\frac{n}{2}r^2\sin\frac{2\pi}{n}$.

16. If the radius of the earth, supposed spherical, be 4000 miles, the length in miles of a degree of longitude in a place the latitude of which is l^0, is $\frac{200}{9}\pi \cos l$.

In the following examples $\pi = \frac{22}{7}$.

17. The driving-wheel of an engine is 5 ft. 3 in. in diameter: how many revolutions does it make in a mile, and at what rate will the train be travelling if it make 3 revolutions per second?

18. Find in yards the length of $1''$ at the equator: radius = 4000 m.

19. Find the number of seconds which an equatorial arc a mile in length subtends at the centre of the earth.

20. A cylinder of iron which supports a bridge is 40 feet high and 7 feet in diameter: find its curved surface, and the number of cubic feet it contains, if the iron is 7 inches thick.

21. The difference between the circumference of a circle and the perimeter of the inscribed regular hexagon is one inch: find the radius.

22. The minute hand of a clock is 3 ft. 9 in. in length: find how far its point will move in 35 minutes.

23. If an arc 1·309 feet long subtend an angle of $7\frac{1}{2}$ degrees at the centre of a circle whose radius is 10 feet, find the ratio of the diameter to the circumference.

24. A railway curve has a radius of a mile and subtends an angle of 75^0 at the centre: how long is it?

25. A railway curve has a radius of 1000 yards, and is 1210 yards long: what angle does it subtend at the centre?

26. A railway curve is a mile long and subtends 60^0 at the centre: find the radius.

27. A train is running on a curve of a mile radius and changes from N. to N.W.: how many yards has it run?

28. What is the length of a rope, 1 inch in diameter, which can be coiled on a drum 3 ft. 6 in. thick, and the same height?

29. Find the radius of a globe such that the distance measured on it between two places on the same meridian whose latitudes differ by $1^0\ 10'$ may be one inch.

30. Find approximately the difference in latitude between two places, one of which is 396 miles north of the other.

31. On a circle, 10 feet in radius, it was found that an angle of $22^0\ 30'$ was subtended by an arc 3 ft. $11\frac{1}{8}$ in. in length: hence find the value of π to four decimal places.

32. Prove that the area of a sector which includes an angle a at the centre is $\frac{a}{2}r^2$, and find the change in this expression when the angle is in degrees.

33. The area of a segment the arc of which subtends an angle $2a$ at the centre is $\left(a - \frac{\sin 2a}{2}\right)r^2$.

34. Two circles are described each of radius r: if the centre of each be on the circumference of the other, find the area included between them.

35. Three circles, each of radius r, touch one another: find the area included between their circumferences.

36. A target 4 feet broad has a bull's-eye of 8 inches in diameter: find in degrees the angles subtended at a distance of 200 yards by the target and by the bull's-eye.

37. A man aims at the centre of a target 500 yards off, but his rifle throws 7″ to the right of his line of aim: how many inches from the centre will the bullet strike the target?

CHAPTER XVII.

TRIGONOMETRICAL EQUATIONS.

74. IN Trigonometrical equations we have given some relation between the ratios of an unknown angle, from which one or more values of one of these ratios can be found, and we have to find *all* the corresponding values of the angle.

We must therefore discover forms for writing down all the values of an angle when one of its ratios is known.

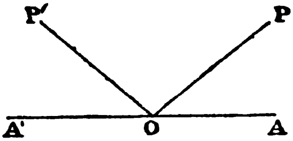
It is convenient to express the solutions in circular measure.

Let α be one angle which has the given ratio, then if the equation be $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$, or $\tan^2\theta = \tan^2\alpha$, we know that the only angles which have the same ratios as α, without regard to sign, are those formed by adding or taking away α from any multiple of two right angles.

The solution is $\therefore\ \theta = n\pi \pm \alpha$.

75. *To find a form for all angles which have the same sine as α.*

Let $AOP = \alpha$, make $A'OP' = AOP$: then $AOP' = \pi - \alpha$.



Then we know (Chap. VI.) that AOP, AOP' and angles which differ from either by some multiple of four right angles are the only angles which have the same sine as α.

The angles which differ from AOP by some multiple of 4 right angles are $2\pi + \alpha$, $4\pi + \alpha \ldots$, $-2\pi + \alpha$, $-4\pi + \alpha \ldots$ which are included in the form $2n\pi + \alpha$, and similarly those which differ from AOP', or $\pi - \alpha$, by the same are included in the form $2n\pi + \pi - \alpha$, or $(2n + 1)\pi - \alpha$.

In these forms the sign of α is + or −, as the coefficient of π is even or odd.

But $(-1)^m = 1$ or -1, as m is even or odd;

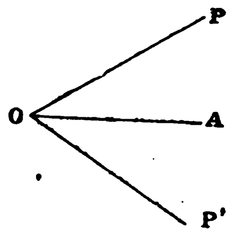
$$\therefore m\pi + (-1)^m \alpha \text{ comprises both cases.}$$

If $\therefore \sin\theta = \sin\alpha$, $\theta = m\pi + (-1)^m \alpha$ where m is integral.

AOP or α need not be an acute angle.

76. *To find a form for all angles which have the same cosine as α.*

Let $AOP = \alpha$, make $AOP' = AOP$; $\therefore AOP' = -\alpha$.

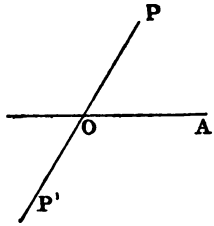


Now we know that the only angles which have the same cosine as α are AOP, AOP', and angles which differ from either by a multiple of four right angles.

These are all included in the form $2m\pi \pm \alpha$.

77. *To find a form for all angles which have the same tangent as α.*

Let $AOP=\alpha$, as before.



Produce PO to P', then AOP, AOP', and all angles formed by adding any multiple of *four* right angles to either, that is, by adding any multiple of *two* right angles to AOP, have the same tangent, and are the only angles that have.

The form is $\therefore\ m\pi+\alpha$.

EXAMPLES:

(i)
$$\sin^2\theta=\tfrac{1}{4},\ \therefore\ \sin\theta=\pm\tfrac{1}{2},$$
$$\text{but } \sin\frac{\pi}{6}=\tfrac{1}{2},\ \therefore\ \theta=n\pi\pm\frac{\pi}{6}.$$

(ii)
$$2\cos^2 x-3\cos x+1=0.$$
Here, either $\cos x=1=\cos 0$, or $\cos x=\tfrac{1}{2}=\cos\dfrac{\pi}{3}$;
$$\therefore\ x=2m\pi \text{ or } 2m\pi\pm\frac{\pi}{3}.$$

(iii)
$$2\sin^2 x=3\cos x.$$
$\therefore\ 2-2\cos^2 x=3\cos x,\ \ 2\cos^2 x+3\cos x=2,\ \ \cos x=\tfrac{1}{2}$ or -2.

The solution -2 is inadmissible, since $\cos x$ must be between 1 and -1.

Also $\cos\dfrac{\pi}{3}=\tfrac{1}{2}$; $\therefore\ x=2m\pi\pm\dfrac{\pi}{3}$.

EXAMPLES ON CHAPTER XVII.

1. Write down the general values of θ, when the following equations are satisfied: $2\sin^2\theta=1$; $4\cos^2\theta=3$; $\sin^2\theta=1$; $\tan^2\theta=3$; $\operatorname{cosec}^2\theta=4$; $\cot^2\theta=3$.

2. Write down the general values of θ for the following values of the sine: $\tfrac{1}{2}$, $-\dfrac{\sqrt{3}}{2}$, 1, -1, $-\dfrac{\sqrt{5}-1}{4}$, $\dfrac{1}{\sqrt{2}}$.

3. Write down the general values of θ for the following values of the cosine: $\tfrac{1}{2}$, $-\dfrac{\sqrt{3}}{2}$, 1, 0, $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $-\dfrac{1}{\sqrt{2}}$.

4. Write down the general values of θ for the following values of the tangent: 1, -1, $\sqrt{3}$, $-\frac{1}{\sqrt{3}}$, $2+\sqrt{3}$, $\sqrt{3}-2$.

5. If $\sin\theta = -\sin\alpha$, $\theta = m\pi + (-1)^{m-1}\alpha$.

6. Shew that the forms $m\pi + (-1)^m\alpha$, and $(4n+1)\frac{\pi}{2} \pm \left(\frac{\pi}{2} - \alpha\right)$ give the same angles.

7. Find, in degrees, all the acute angles which satisfy the equation $\sin 10\theta = \frac{1}{2}$.

8. Find, in degrees, all the obtuse angles which satisfy $\cos 12\theta = \frac{1}{2}$.

9. Find, in degrees, all the angles between 180^0 and 270^0, for which $\tan 5\theta = 1$.

Solve the following equations:

10. $\cos^2\theta - \sin^2\theta = \frac{1}{2}$. 11. $\cos\theta + \sin\theta = \frac{1}{\sqrt{2}}$. 12. $\sin 2\theta = \sin\theta$.

13. $\tan\frac{\theta}{2} = \operatorname{cosec}\theta - \sin\theta$. 14. $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.

15. $2\sin^2 3\theta + \sin^2 6\theta = 2$. 16. $4\sin^2\theta + \sin^2 2\theta = 3$. 17. $\cos 3\theta = \cos\theta$.

18. $\tan\theta + 3\cot\theta = 4$. 19. $2\sin\theta - \sin 2\theta = (1+\cos\theta)^2$.

20. $\tan^{-1}\frac{x+1}{x-1} - \tan^{-1}\frac{x-1}{x+1} = \tan^{-1}\frac{1}{2}$. 21. $\cos 2\theta + \cos 4\theta = 2\cos 3\theta$.

22. $\tan\theta\sin\theta = \frac{3}{2}$. 23. $\sin 2x = \cos 3x$.

24. $\cos 3x + \sin 3x = \cos x + \sin x$. 25. $\tan^2 x + 4\sin^2 x = 6$.

26. $2\tan\theta + 4 = (2+\sqrt{3})\sec\theta$. 27. $1 + \sin x = \frac{3}{4}\cos x + \frac{3}{4}\tan x$.

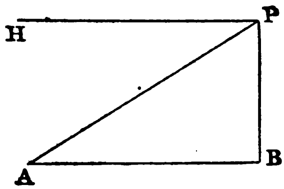
CHAPTER XVIII.

PROBLEMS.

78. THE principal applications of Trigonometry are in measuring heights and distances.

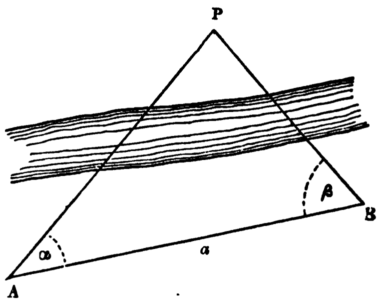
By using a sextant, an observer can measure the angle subtended at his eye by any two objects which he can see.

By means of a theodolite, he can measure any angles in a vertical or horizontal plane.



If P be any object above the horizontal plane, PAB the angle which PA makes with the horizontal plane through A, then PAB is called the *angle of elevation of P.* Similarly, if A be observed from P the angle APH, which PA makes with the horizontal plane through P, is called the *angle of depression of A.*

79. A, B *are two stations on a horizontal plane, from which a third* P *can be seen, to find the distance of* P *from each.*



Let $AB = c$, $PAB = \alpha$, $PBA = \beta$.

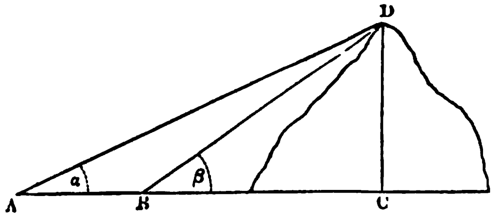
By Art. 40, $AP = AB\dfrac{\sin PBA}{\sin APB} = c\dfrac{\sin\beta}{\sin(\alpha+\beta)}$;

since $$APB = 180^\circ - \alpha - \beta.$$

Similarly $$BP = c\frac{\sin\alpha}{\sin(\alpha+\beta)}.$$

80. *An object, above an horizontal plane, is observed from two stations on the horizontal plane, to find its height.*

Let D be the highest point of the object, A, B the two stations: let $AB = a$.



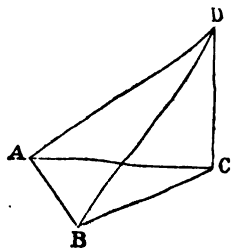
(i) Let ABD be a vertical plane, and let the angles of elevation from A and B be α, β respectively; let $CD = x$.

Then $x = AD \sin \alpha$,

$$\text{but} \quad AD = AB \frac{\sin \beta}{\sin ADB} = a \frac{\sin \beta}{\sin (\beta - \alpha)};$$

$$\therefore\ x = a \frac{\sin \alpha \sin \beta}{\sin (\beta - \alpha)}.$$

(ii) Next let A and B not be in the same vertical plane with D.



As before, let the angles of elevation of D at A and B be α and β.

Measure DAB, DBA, and denote them by γ and δ respectively.

Then

$$DA = AB \frac{\sin DBA}{\sin ADB} = a \frac{\sin \delta}{\sin (\gamma + \delta)},$$

$$DC = DA \sin \alpha = a \frac{\sin \delta \sin \alpha}{\sin (\gamma + \delta)}.$$

Exactly in the same way we might have found

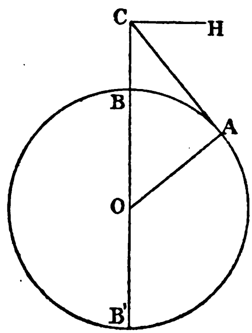
$$DC = a\frac{\sin\gamma\sin\beta}{\sin(\gamma+\delta)}.$$

The agreement of the two results will give us a test of the accuracy of our observations.

81. The surface of the earth is not really plane but that of a spheroid, or approximately a sphere of 4000 miles radius.

Calculations made on the supposition that the earth is flat are necessarily erroneous, and though the error is inconsiderable if the stations are close together it cannot be neglected if they are far apart.

82. *To find a relation between the height of an object, and the distance at which it can just be seen from the surface of the earth, supposing the earth to be a sphere of* 4000 *miles radius.*



Let A be a point on the earth's surface from which an object BC of height h *feet* can just be seen.

Let O be the centre of the earth, $B'OB$ that diameter which, when produced, passes through C.

Let $AB = d$ *miles*, $OA = r$ *miles*, where $r = 4000$ nearly.

Then the plane in which CB, CA are must pass through the centre O. Let $BOA = \theta$, then (Art. 69) $BA = r\theta$, $CA = r\tan\theta$.

Since θ is small, we may consider θ and $\tan\theta$ to be equal (Art. 72) and $\therefore r\theta = r\tan\theta$, nearly.

Now AC is the tangent at A;

$$\therefore AC^2 = CB \,.\, CB' \text{ (Euc. III. 36)}$$

or $$d^2 = \frac{h}{5280}\left(2r + \frac{h}{5280}\right) = \frac{800}{528}h + \frac{h^2}{5280^2}.$$

If h be small, $\left(\frac{h}{5280}\right)^2$ is a small fraction, and can be neglected in comparison with h, and then we have

$$d^2 = \frac{800}{528}h = \tfrac{3}{2}h, \text{ nearly.}$$

This gives the ordinary rule—*the height of an object in feet is equal to two-thirds of the square of the distance, at which it can be seen, in miles.*

83. *Dip of the horizon.*

Let CH (in the preceding figure) be the horizontal line through C in the plane OCA: it is obvious that as CB increases, CA and HCA increase too.

The angle HCA is called the dip of the horizon from C: for since A is obviously a point on the bounding line of that portion of the earth's surface which can be seen from A, A is on the horizon.

Let $HCA = \theta$, $CB = h$ (in ft.), $BA = d$ (in miles).

Then since $HCA = COA$, (each being the complement of ACO), $\theta = \frac{d}{r} = \frac{d}{4000}$; but $d^2 = \frac{3}{2}h$.

Hence by substitution we can get θ in terms of h.

Or thus; if h be the height *in miles*, r the radius,.

$$\cos\theta = \frac{AO}{OC} = \frac{r}{r+h} = \left(1+\frac{h}{r}\right)^{-1};$$

$$\therefore \left(1-\sin^2\theta\right)^{\frac{1}{2}} = \left(1+\frac{h}{r}\right)^{-1};$$

$\therefore \frac{1}{2}\sin^2\theta = \frac{h}{r}$, nearly, neglecting powers of $\sin\theta$ and of $\frac{h}{r}$ higher than the first.

Since θ is small, $\sin\theta = \theta$ nearly, $\therefore \theta^2 = \frac{2h}{r}$.

θ is here the circular measure of the angle. To obtain the angle in degrees we must multiply by $\frac{180}{\pi}$.

EXAMPLES ON CHAPTER XVIII.

1. A man stands at a point A on the bank AB of a straight river, and observes that the line joining A to a post C on the opposite side makes an angle of 30^0 with AB. He then walks 400 yards along the bank to B, and finds the angle ABC to be 60^0. Shew that the breadth of the river is $100\sqrt{3}$ yards.

2. An isosceles triangle of wood is placed in a vertical plane, vertex upwards, facing the sun. If $2a$ be the base of the triangle, h its height, 30^0 the altitude of the sun, the tangent of the angle at the apex of the shadow is $\dfrac{2ah\sqrt{3}}{3h^2-a^2}$.

3. The length of the shadow of a vertical stick is to the length of the stick as $\sqrt{3} : 1$. If the stick be turned about its lower end in a vertical plane so that the shadow is in the same direction, find its inclination to the horizon when the length of the shadow is the same as before.

4. Standing opposite the stern of a barge which is moored parallel to the bank of a stream, I find its length subtends an angle of 45^0; walking 100 feet along the bank, so that I pass by its bow, I find its length subtends an angle whose tangent is $\frac{1}{7}$. Shew that the barge is fifty feet long.

5. A target is six feet in height and twelve in breadth: find the tangents of the angles its four sides subtend at a point 800 yards in front of its bottom right-hand corner.

6. The base of a pyramid is square, and its faces equilateral triangles: find the inclinations of the faces to the base.

7. An object six feet high, placed on the top of a tower, subtends an angle, the tangent of which is ·015, at a place the horizontal distance of which from the foot of the tower is 100 feet: find the height of the tower.

8. The angles of elevation of a balloon (α, β, γ) were taken at the same time by three observers, one at each end and one at the middle point of a base of length $2a$ on a horizontal plane. Find the height of the balloon.

9. A person wishing to ascertain the height of a rock on a plane inclined at an angle of 15^0 to the horizon, ascends 80 feet in a vertical plane with the rock, and then it subtends an angle of 30^0; find its height.

10. The elevation of a steeple on a horizontal plane is observed, and at a station a feet nearer its elevation is found to be the complement of the first; on advancing b feet nearer still, the elevation is double the first: shew that the height is

$$\left\{(a+b)^2 - \frac{a^2}{4}\right\}^{\frac{1}{2}}.$$

11. An object above a horizontal plane is observed from two stations A, B, c feet apart and in the same vertical plane with it; its angles of elevation being α, $\frac{\pi}{2}-\alpha$: determine its height.

12. If the angles in the preceding question be α, 2α, determine the height.

13. Solve the two preceding questions numerically, when $c=100$, $\alpha=22^0\ 30'$.

14. From a horizontal road running E. and W. a mountain is observed due S. at an angle of elevation α; after walking a mile along the road its elevation is β: determine its height and distance from the road as the crow flies.

15. A tower stands on a horizontal plane, and a flagstaff at the top is found to subtend the same angle, 15^0, at two points on the plane, whose distances from the foot of the tower are a and b feet. Shew that the height of the flagstaff is $\frac{a+b}{2+\sqrt{3}}$ feet.

16. From the top of a cliff 100 yards high, the angles of depression of the top and bottom of a vertical river-bank opposite are $38^0\ 39'\ 36''$ and $51^0\ 20'\ 24''$. Find the breadth of the river, and the height of the bank. Given

$\log 2=\cdot 30103$, $\quad \text{L}\tan 38^0\ 39'=9\cdot 902928$, $\quad \text{L}\tan 38^0\ 40'=9\cdot 903198$.

17. A person in a ship whose course is due north observes that two fixed objects, known to be a mile apart and east and west of each other, are in a straight line: how many yards will the ship have gone when it is observed that the two objects bear angles of $65^0\ 24'$ and $52^0\ 5'$ with the course? Given

$\text{L}\cos 65^0\ 24'=9\cdot 319386$, $\quad \text{L}\sin 13^0\ 19'=9\cdot 362356$, $\quad \log 1\cdot 95=\cdot 292035$,

$\text{L}\cos 52^0\ 5'=9\cdot 788532$, $\quad \log 176=2\cdot 245513$, $\quad \log 1\cdot 96=\cdot 292256$.

18. From two points A and B in the same vertical plane with a tower, and 66 feet apart on a horizontal plane, the elevations of the tower are $29^0\ 30'$, $21^0\ 20'$ respectively: find its height. Given

$\text{L}\sin 29^0\ 30'=9\cdot 6923388$, $\quad \text{L}\sin 8^0\ 10'=9\cdot 1524507$, $\quad \log 8\cdot 3231=\cdot 9202851$,

$\text{L}\sin 21^0\ 20'=9\cdot 5608546$, $\quad \log 66=1\cdot 8195439$, $\quad \log 8\cdot 3232=\cdot 9202903$.

19. A person standing at a point A, due south of a tower built on a horizontal plane, observes the altitude to be 60^0. He then walks to a point B, due west from A, and observes the altitude to be 45^0, and again at C in AB produced, he observes the altitude to be 30^0. Shew that $AB=BC$.

20. A man stands on a wall height h, and observes the elevation (α) of the top of a telegraph post: he then descends from the wall and observes the elevation (β). Shew that the height of the post exceeds that of the man by $\dfrac{h \sin\beta \cos\alpha}{\sin(\beta - \alpha)}$.

21. The angular elevation of a tower at a place due north of it is 45°, and at another place due east of the former 15°. Shew that the height of the tower is $\frac{a}{2}(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$, where a is the distance between the two places.

22. The altitude of a certain rock is observed to be 47°, and after walking towards it 1000 feet up a slope inclined at 32° to the horizon the altitude is 77°. Find the vertical height of the rock above the first point of observation. Given $\sin 47° = \cdot 73135$.

23. An observer on the top of a cliff, 200 feet above the sea level, notices that the angles of depression of two boats at anchor in the same vertical plane with him are 30° and 45°. Find the distance of the boats from each other.

24. A tower stood at the foot of a plane inclined at 9° to the horizon; a line 100 feet in length was measured straight up the incline from the foot of the tower; at the upper end of this line the tower subtended an angle of 54°: find its height.

25. The lamp of a lighthouse is 216 feet above the sea: how far off can it be seen?

26. Lighthouses are placed at intervals of 16 miles along a straight coast: at what height must the light be fixed so that one at least may be visible at a distance of six miles from any point of the coast?

27. A boat has a light six feet above the sea: at what distance will it just be visible from the deck of a ship 24 feet above the sea?

28. What must be the height of the mast of a ship which is just visible at a distance of 15 miles from the deck of another ship 8 ft. 2 in. above the sea?

29. Supposing the radius of the earth to be 4000 miles, find the dip of the horizon from a balloon four miles from its surface, and the height of the balloon, when the dip is one degree. ($\pi = \frac{22}{7}$.)

30. If a light on a pole 33 feet high be just visible at a distance of seven miles, find the earth's radius. ($\pi = \frac{22}{7}$.)

MISCELLANEOUS EXAMPLES.

1. ABC is a triangle, and the measures of A, B, C, in degrees, grades and circular units are as $1 : 1 : \frac{120}{\pi}$: express each in degrees.

2. Express the following in terms of ratios of angles less than half a right angle: $2 \sin 510^0 + \tan 703^0 \tan 253^0 - \operatorname{cosec} 236^0$.

3. $\cos A = \cos^4 \frac{A}{2} - \sin^4 \frac{A}{2}$.

4. If $\cos A \cos 3A = \frac{5}{18}$, then $\tan A = \pm \frac{1}{\sqrt{5}}$.

5. $\sin \alpha + \sin \beta - \cos \alpha \sin (\alpha + \beta) = 2 \sin \alpha \sin^2 \frac{\alpha + \beta}{2}$.

6. If $\log 2 = \cdot 30103$, and $\log 3 = \cdot 4771213$, find $\log \left(\frac{729}{1944}\right)^{\frac{5}{81}}$.

7. The angles of a triangle are as $1 : 2 : 3$, and the perpendicular from the greatest angle on the opposite side is p: find the area.

8. Eliminate θ from the equations,

$$x \cos \theta + y \sin \theta = a, \quad x \sin \theta - y \cos \theta = b.$$

9. Solve $\sin 5\theta = \sin 3\theta$.

10. ABC is a triangle: AD, BE, which are perpendicular to BC, CA respectively, intersect in O: shew that $OA = a \cos A$.

11. Find the French measure and the circular measure of $25^0\ 2'\ 25''$.

12. If the radius of the earth be 4000 miles, find the difference in latitude of two places, one of which is 200 miles south of the other.

13. If $1 + \sin^2 \theta = 3 \sin \theta \cos \theta$, find $\tan \theta$.

14. $\cos (45^0 - A) \cos (45^0 - B) - \cos (45^0 + A) \cos (45^0 + B) = \sin (A + B)$.

15. Given $\log_8 9 = a$, and $\log_3 5 = b$: find $\log_{10} 2$ and $\log_3 10$ in terms of a and b.

16. $(b - c)(s - a) \cos^2 \frac{A}{2} + (c - a)(s - b) \cos^2 \frac{B}{2} + (a - b)(s - c) \cos^2 \frac{C}{2} = 0$.

17. Eliminate θ from $x \cos \theta + y \sin \theta = a = x \sin \theta - y \cos \theta$.

18. If

$$a \sin A + b \sin B + c \sin C = 0,$$
$$a \cos A + b \cos B + c \cos C = 0,$$

then

$$\frac{\sin (B - C)}{a} = \frac{\sin (C - A)}{b} = \frac{\sin (A - B)}{c}.$$

19. Solve $\dfrac{\sin^4\theta}{\sin^2\alpha}+\dfrac{\cos^4\theta}{\cos^2\alpha}=1.$

20. If D be the middle point of the side BC of a triangle ABC, and S be the area, then $\cot ADB=\dfrac{AC^2-AB^2}{4S}$.

21. The angles of a triangle are in A. P., and the number of grades in the least = number of degrees in the greatest: find them in circular measure.

22. Calculate the distance at which a globe of $5\frac{1}{2}$ inches in diameter will subtend an angle of $6'$ $(\pi=\frac{22}{7})$.

23. Trace the changes in sign and value of $\cos\theta+\sin\theta$, as θ changes from 0 to 2π.

24. If
$$\tan\theta=\frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha},$$
then
$$\tan\frac{\theta}{2}=\frac{(1-\sin\beta)(1-\cos\alpha)}{\sin\alpha\cos\beta},$$
and
$$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{\alpha-\beta}{2}\right)}.$$

25. $\tan(A+B)\tan(A-B)=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}$.

26. If $\sin A=m$, $\sin\dfrac{A}{2}$ has four values, whose sum is zero, and product $\dfrac{m^2}{4}$.

27. If two sides of a triangle 4 and 5 and the cosine of the difference of the opposite angles be $\frac{31}{32}$, then the third side will be 6.

28. $ABCD$ is a quadrilateral, whose opposite sides AB, CD are at right angles: shew that $\cot DAB \cot ABC=\cot CAB \cot ABD$.

29. Solve $\tan^2\theta=3\operatorname{cosec}^2\theta-1$.

30. If circles can be inscribed in and described about a quadrilateral, whose consecutive sides are a, b, c, d: shew that the diagonals meet at an angle $\cos^{-1}\dfrac{ac-bd}{ac+bd}$.

31. The angles of a triangle are in A. P., and the number of degrees in the least is to the number of grades in the greatest as 3 : 5: find them in degrees.

32. The large wheel of a bicycle is 56 inches in diameter, and the small wheel 14 inches: how many turns will each make in a mile? $\left(\pi=\dfrac{22}{7}.\right)$

33. Trace the changes in sign and value of $\tan\theta - \cot\theta$, as θ changes from 0 to π.

34. $\dfrac{\sin 3\theta + 2\sin 5\theta + \sin 7\theta}{\sin 5\theta + 2\sin 7\theta + \sin 9\theta} = \dfrac{\sin 5\theta}{\sin 7\theta}$.

35. $\tan\theta = 4\tan\dfrac{\theta}{4}\,\dfrac{1 - \tan^2\dfrac{\theta}{4}}{1 - 6\tan^2\dfrac{\theta}{4} + \tan^4\dfrac{\theta}{4}}$.

36. Given $\log_{10} 2 = 30103, \quad \log_{10} 7 = \cdot 845098$:
find $\log_{10} 98$ and $\log_{1000} \frac{343}{8}$.

37. $\dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca} = \dfrac{1}{2Rr}$.

38. If $m = \operatorname{cosec}\theta - \sin\theta, \quad n = \sec\theta - \cos\theta,$
then $m^{\frac{2}{3}} n^{\frac{2}{3}} (m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1.$

39. Solve the equation $\sec\theta - 2\cos\theta = 1$.

40. If the distance between two stations A, B on a horizontal plane subtend equal angles (α) at two other stations C and D on the plane, and if β be the sum of the angles CDB, DCA, then

$$AB = CD\,\frac{\sin\alpha}{\sin(\beta - \alpha)}.$$

41. ABC is a triangle, right-angled at C, and the number of degrees in A : circular measure of B :: 45 : π: find A and B.

42. The driving wheel of an express engine is six feet in diameter, and makes 280 revolutions per minute: what is the rate?

43. If $\cos A = \dfrac{4mn}{m^2 + n^2}$, find $\operatorname{cosec} A$.

44. $\sec A + \sec(120^\circ + A) + \sec(120^\circ - A) + 3\sec 3A = 0.$

45. If $\dfrac{\sin(\alpha - \beta)}{\sin\beta} = \dfrac{\sin(\alpha + \theta)}{\sin\theta}$, then $\cot\beta - \cot\theta = \cot(\alpha + \theta) + \cot(\alpha - \beta)$.

46. $\dfrac{a\cos B - b\cos A}{c} = \dfrac{\sin^2 A - \sin^2 B}{\sin^2 C}$.

47. Two sides of a triangle are a and $2a$, and they include a right angle; if r_1, r_2 be the radii of the escribed circles touching these sides $r_1 r_2 = a^2$.

48. Eliminate θ and ϕ from

$$a\sin^2\theta + b\cos^2\theta = m,$$
$$b\sin^2\phi + a\cos^2\phi = n,$$
$$a\tan\theta = b\tan\phi.$$

49. Solve the equation $2\sin\theta\sin 3\theta = \sin^2 2\theta$.

50. If θ be the acute angle between two adjacent sides of a rhombus, whose diagonals are x and y, then

$$\cos\theta = \frac{x^2 - y^2}{x^2 + y^2}.$$

51. If l be the length of the arc which subtends an angle θ at the centre of a circle of radius r, and if $\theta = 3\frac{l}{r}$, what is the unit?

52. $(\operatorname{cosec}^2 A - 1)(2 \operatorname{vers} A - \operatorname{vers}^2 A) = \cos^2 A$.

53. $\cos 3A - \sin 3A = (\cos A + \sin A)(1 - 2\sin 2A)$.

54. $\cos\frac{6\pi}{25} + \cos\frac{7\pi}{25} + \cos\frac{8\pi}{25} + \cos\frac{9\pi}{25} = 4\sin\frac{\pi}{5}\cos\frac{\pi}{25}\cos\frac{\pi}{50}$.

55. $$\tan^{-1}\tfrac{4}{3} = \tfrac{1}{2}\tan^{-1}(-\tfrac{24}{7}) = \tfrac{1}{3}\cos^{-1}(-\tfrac{117}{125}).$$

56. If in the triangle ABC, $\tan\theta = \frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}$, then $a = (b-c)\sec\theta$.

57. If $\frac{\tan A - \tan B}{\tan A + \tan B} = \frac{c-b}{c}$, then $A = 60^\circ$.

58. If

$$x\cos\beta + y\cos\alpha = z,$$
$$x\sin\beta - y\sin\alpha = 0,$$

then

$$\frac{x}{\sin\alpha} = \frac{y}{\sin\beta} = \frac{z}{\sin(\alpha+\beta)}.$$

59. If $\cos p\theta + \cos q\theta = 0$, prove that the different values of θ form arithmetic progressions, of which the common differences are $\frac{2\pi}{p+q}$ and $\frac{2\pi}{p-q}$ respectively.

60. If Δ be the area of a quadrilateral inscribed in a circle, shew that the diagonals meet at the angle $\sin^{-1}\frac{2\Delta}{ac+bd}$, where a, b, c, d are consecutive sides.

61. If $\theta = \frac{18}{\pi} \cdot \frac{l}{r}$, where l is the length of the arc which subtends an angle θ at the centre of a circle of radius r, what is the unit?

62. AB, AC are two radii of a circle at right angles to one another, on BC as diameter a circle is described: find the area of the part outside the quadrantal arc and within the semicircle ($AB = a$).

63. $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$.

64. $\left(\frac{1+\cos\alpha}{1-\cos\alpha} - \frac{\sec\alpha - 1}{1+\sec\alpha} - 4\cot^2\alpha\right)(\sec\alpha + 1) = 4$.

65. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2} = 120^\circ$.

66. $\tan\frac{3A}{2} = \frac{\sin 3B - \sin 3C}{\cos 3C - \cos 3B}$, where ABC is a triangle.

67. In an isosceles right-angled triangle a straight line is drawn from the middle point of one side to the opposite angle: shew that it divides that angle into two parts, whose cotangents are 2 and 3.

68. If O be the centre of the inscribed circle of a triangle, then
$$a \cdot AO^2 + b \cdot BO^2 + c \cdot CO^2 = abc.$$

69. Solve $2\tan^{-1}(\cos\theta) = \tan^{-1}(2\operatorname{cosec}\theta)$.

70. Two towers, whose heights are 80 and 180 feet respectively, stand on a horizontal plane; from the foot of each the angle of elevation of the other is taken, and one is double the other: shew that the horizontal distance between the towers is 240 feet.

71. The angles of a triangle are as $5 : 7 : 8$: express them in each of the three measures.

72. Find the areas between the circumference of a circle and its inscribed and circumscribing squares.

73. If $\cot A = 1 + \sqrt{2}$, find $\cos A$.

74. $\sin 18^\circ + \sin 30^\circ = \sin 54^\circ$.

75. $\tan^{-1}(\cot A) + \tan^{-1}(\tan A) = n\pi + \dfrac{\pi}{2} - 2A$.

76. $\tan\dfrac{15^\circ}{2} = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)$.

77. In a triangle $A = 90^\circ$ and $B = 60^\circ$, shew that $2a = c + b\sqrt{3}$.

78. $r_1 \cot\dfrac{A}{2} = r_2 \cot\dfrac{B}{2} = r_3 \cot\dfrac{C}{2} = r\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$.

79. Solve $\operatorname{cosec}\theta + \cot\theta = \sqrt{3}$.

80. Regular polygons of the same number of sides are inscribed in and described about a circle. If their areas be as $3 : 4$, find the number of sides.

81. Find the unit when 120° is represented by $\frac{1}{1}\frac{5}{6}$.

82. The circle inscribed in an equilateral triangle is cut out: compare the area left with that of the triangle.

83. Trace the changes in sign and value of $\sec\theta - \cos\theta$ as θ changes from 0 to 2π.

84. $\cos 30^\circ + \cos 60^\circ + \cos 210^\circ + \cos 270^\circ = \frac{1}{2}$.

85. $2\cot^{-1} x = \operatorname{cosec}^{-1}\dfrac{1+x^2}{2x}$.

86. If a, b, c be in A. P., then $2\sin\dfrac{A}{2}\sin\dfrac{C}{2} = \sin\dfrac{B}{2}$.

87. The points where the circle inscribed in a triangle ABC touches the sides are joined. Shew that the sides of the triangle so formed are proportional to $\cos\dfrac{A}{2}$, $\cos\dfrac{B}{2}$, $\cos\dfrac{C}{2}$ respectively.

88. If $a\cos\theta+b\sin\theta=a\cos\phi+b\sin\phi=c$, and $\tan\theta\tan\phi=m$, then
$$c^2-a^2=m(c^2-b^2).$$

89. Solve $\cos 8\theta-\cos 5\theta+\cos 3\theta=1$.

90. If the mast of a ship, the top of which is 96 feet above the water, be just visible at a distance of twelve miles, determine the earth's radius approximately.

91. The unit angle is 15^0 and measure of an angle $=k\times\frac{\text{arc}}{\text{rad}}$: what is the value of k?

92. Trace the changes in sign and value of $\cos\pi(\cos\theta-\sin\theta)$ as θ changes from 0 to 2π.

93. $\sec^2\frac{\pi-a}{4}+\sec^2\frac{\pi+a}{4}=\sec^2\frac{\pi-a}{4}\sec^2\frac{\pi+a}{4}$.

94. $\frac{1-\sin a\cos a}{\cos a(\sec a-\operatorname{cosec} a)}\times\frac{\sin^2 a-\cos^2 a}{\sin^3 a+\cos^3 a}=\sin a$.

95. $\frac{\cos 40^0}{\cos 80^0}-\frac{\cos 80^0}{\cos 20^0}-\frac{\cos 20^0}{\cos 40^0}=3$.

96. Find the area of a triangle whose three sides are ·9, 1·2, 1·5.

97. $\frac{r}{R}=4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right)$.

98. Eliminate θ and ϕ from the equations
$$y=x\tan\theta+a\cot\theta,\quad y=x\tan\phi+a\cot\phi,\quad \tan\theta+\tan\phi=m.$$

99. If α,β,γ be the perpendiculars from the angles on the sides of ABC, then $a\sin A+b\sin B+c\sin C=2(\alpha\cos A+\beta\cos B+\gamma\cos C)$.

100. From a balloon at the height of one mile the dip of the horizon is observed to be $1^0\ 16'\ 20''$: determine the earth's radius.

101. In the equation, angle$=k\cdot\frac{\text{arc}}{\text{rad}}$, shew that $\frac{1}{k}$ is the circular measure of the unit of angular measurement.

102. What is the circular measure of the angle between the hands of a watch at a quarter to twelve o'clock?

103. If $\tan a=n+\frac{n}{2n+1}$, find $\operatorname{cosec} a$ and $\operatorname{vers} a$.

104. $\cot a-\tan a=2\operatorname{cosec}4a+2\cot 4a$.

105. If A, B, C are in A.P., then $\frac{\sin A-\sin C}{\cos C-\cos A}=\cot B$.

106. In any triangle ABC $\log a=\log(b-c)+\mathrm{L}\cos\frac{A}{2}-\mathrm{L}\cos\phi$, where
$$\tan\phi=\frac{b+c}{b-c}\tan\frac{A}{2}.$$

107. If ABC be a triangle, and $C=90^0$, then $\tan^{-1}\frac{a}{b+c}+\tan^{-1}\frac{b}{a+c}=45^0$.

108. If the radius of the circumscribing circle of a triangle be equal to the diameter of the inscribed circle, the triangle is equilateral.

109. A flagstaff six feet high on the top of a wall subtends an angle 15^0 at a point on a horizontal plane on which the wall stands: the elevation of the wall being 45^0, find its height.

110. The straight lines bisecting the angles of the triangle ABC meet the opposite sides in L, M, N respectively: shew that

$$AL = \frac{2S}{(b+c)\sin\frac{A}{2}},$$

that the area of LCM is $\dfrac{abS}{(a+c)(b+c)}$,

and that of LMN $\dfrac{2abcS}{(a+b)(b+c)(c+a)}$,

where S is the area of ABC.

111. The number of grades in an angle of a regular polygon exceeds the number of degrees by 16: how many sides has the polygon?

112. The diameter of the moon is 2160 miles: if it subtends an angle of $31'$ at the surface of the earth, what is its approximate distance from the earth?

113. $\dfrac{1+\tan\alpha}{\sec\alpha} - \dfrac{1}{\sin\alpha+\cos\alpha} = \dfrac{2}{\sec\alpha+\operatorname{cosec}\alpha}$.

114. If $\tan\theta = \tan^3\dfrac{\phi}{2}$, and $3\cos^2\phi = m^2 - 1$,

then $$\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta = \left(\frac{2}{m}\right)^{\frac{1}{3}}.$$

115. Given $a = 2b$, $A = 3B$: find A, B, C and c.

116. Find the sine of the greatest angle of the triangle whose sides are 11, 13, 15.

117. ABC is a triangle, D, E, F the points where the perpendiculars from the angles meet the sides: shew that $DE = R\sin 2A$, and find the area of DEF.

118. If $ay\cos\alpha = bz\cos\beta$, $x+z=y$, $ay\sin\alpha - bz\sin\beta = y^2 - z^2$,

then $$x = \frac{ab\sin(\alpha-\beta)}{a\cos\alpha + b\cos\beta}.$$

119. The product of the perpendiculars from the angular points on the sides of the triangle ABC is $\dfrac{(a+b+c)^3 r^3}{abc}$.

120. If O be the centre of the inscribed, P, Q, R of the escribed circles of a triangle, prove that

$$OP^2 + QR^2 = OQ^2 + RP^2 = OR^2 + PQ^2.$$

ANSWERS.

EXAMPLES ON CHAPTER I, PAGE 4.

1. $66^g\ 66^{\backprime}\ 66^{\backprime\backprime}\cdot\dot{6}$, $63^g\ 88^{\backprime}\ 88^{\backprime\backprime}\ \dot{8}$, $50^g\ 61^{\backprime}\ 20^{\backprime\backprime}\cdot 7$.

2. $58^0\ 46'\ 52''\cdot 5$, $157^0\ 19'\ 45''$, &c., $4^0\ 32'\ 42''\cdot 62$.

3. 870^0. **4.** 20 minutes to 5, or $3\frac{7}{11}$ minutes past 4.

5. 36^0, 72^0, 72^0. **6.** $\frac{60}{x}$. **7.** $\frac{10b}{9x}$. **8.** $\frac{10y}{9x}$.

9. 180. **10.** $20'$. **11.** 108^0, 120^0, 144^0.

13. 36. **14.** 12, 6. **15.** 10, 8. **16.** 6, 3.

17. 90^0, 60^0, 30^0. **18.** $C=135^0$, $D=105^0$.

EXAMPLES ON CHAPTER II, PAGE 7.

1. $\sin A=\frac{BC}{AB}=\frac{CD}{AC}=\frac{BD}{BC}$: each ratio can be expressed in 3 ways.

2. 45. **3.** 30^0. **4.** 108^0. **5.** $ATB=180^0-A$, $TAB=TBA=\frac{A}{2}$.

6. 120^0, 30^0, 30^0. **7.** See Euc. VI. 3.

EXAMPLES ON CHAPTER III, PAGE 11.

1. $\sin A=\frac{4}{5}$, $\cot A=\frac{3}{5}$, $\cot A=\frac{3}{4}$, $\sec A=\frac{5}{3}$, $\operatorname{cosec} A=\frac{5}{4}$.

2. $\cos A=\frac{5}{13}$, $\tan A=\frac{12}{5}$. **3.** $\frac{9}{41}$, $\frac{40}{9}$. **4.** $\frac{5}{12}$.

5. $\frac{1}{61}$, $\frac{11}{61}$. **6.** $3\frac{4}{7}$.

7. $\sin A=\sqrt{1-\cos^2 A}$, $\tan A=\frac{\sqrt{1-\cos^2 A}}{\cos A}$, &c.

$$\sin A=\frac{\tan A}{\sqrt{1+\tan^2 A}},\ \cos A=\frac{1}{\sqrt{1+\tan^2 A}},\ \text{\&c.}$$

EXAMPLES ON CHAPTER IV, PAGE 15.

8. The triangle is equilateral. 9. 60°, 30°, 90°.

10. $\cot 18^\circ = \sqrt{5+2\sqrt{5}}$, $\tan 18^\circ = \dfrac{\sqrt{5-2\sqrt{5}}}{\sqrt{5}}$, $\sec 72^\circ = \operatorname{cosec} 18^\circ = \sqrt{5}+1$.

11. $A=60^\circ$, $B=36^\circ$. 12. $A=30^\circ$, $B=45^\circ$.

13. $A=45^\circ$, $B=30^\circ$. 14. 30°. 15. 45°, 30°. 16. 30°.

EXAMPLES ON CHAPTER VI, PAGE 23.

2. $\sin(270^\circ+A) = -\cos A$, $\cos(270^\circ+A) = \sin A$, $\tan(270^\circ+A) = -\cot A$.
$\sin(270^\circ-A) = -\cos A$, $\cos(270^\circ-A) = -\sin A$, $\tan(270^\circ-A) = \cot A$.

3. The sines and cosines are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$; $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

4. $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, $\frac{1}{2}$, 1, $\frac{1}{\sqrt{2}}$.

5. $-\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $-\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, $-\sqrt{3}$, 1, -1, $\frac{1}{\sqrt{3}}$, $-\sqrt{3}$.

6. $\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $-\sqrt{3}$, $-\frac{1}{\sqrt{3}}$, $-\sec 15^\circ$, $\sqrt{2}$, $\frac{1}{2}$.

7. $\cos 31^\circ$, $\sin 44^\circ$, $\tan 9^\circ$, $\cot 15^\circ$, $\sec 45^\circ$, $-\sec 22^\circ$, $\operatorname{vers} 18^\circ$.

EXAMPLES ON CHAPTER VIII, PAGE 31.

1. $\sin 15^\circ = \cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $\cos 15^\circ = \sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$,
$\tan 15^\circ = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$, $\tan 75^\circ = 2+\sqrt{3}$, $\sin 105^\circ = \sin 75^\circ$,
$\sin 165^\circ = \sin 15^\circ$, $\cos 105^\circ = -\cos 75^\circ$, $\cos 165^\circ = -\cos 15^\circ$.

2. $\sin 36^\circ = \cos 54^\circ = \frac{1}{4}\sqrt{10-2\sqrt{5}}$, $\cos 36^\circ = \sin 54^\circ = \dfrac{\sqrt{5}+1}{4}$,
$\tan 36^\circ = \frac{1}{2}\sqrt{10+2\sqrt{5}}$, $\tan 54^\circ = \cot 36^\circ$.

6. $\frac{56}{65}$, $\frac{33}{65}$. 7. (i) $\frac{156}{205}$, $\frac{84}{187}$; (ii) $\frac{84}{205}$, $-\frac{156}{133}$.

EXAMPLES ON CHAPTER X, PAGE 40.

1. $\sin 22^\circ 30' = \frac{1}{2}\sqrt{2-\sqrt{2}}$, $\cos 22^\circ 30' = \frac{1}{2}\sqrt{2+\sqrt{2}}$, $\tan 22^\circ 30' = \sqrt{2}-1$, $\sin 112^\circ 30' = \cos 22^\circ 30'$, $\cos 112^\circ 30' = -\sin 22^\circ 30'$.

2. $\tan 7\frac{1}{2}^\circ = \sqrt{6}+\sqrt{2}-\sqrt{3}-2$, $\cot 9^\circ = \sqrt{5}+1+\sqrt{5+2\sqrt{5}}$.

3. $\sin 3^\circ = \sin(18^\circ - 15^\circ)$, $\sin 6^\circ = 2\sin 3^\circ \cos 3^\circ$, $\sin 21^\circ = \sin(18^\circ + 3^\circ)$.

8. $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$. **15.** $A < 45^\circ$ or $> 225^\circ$.

18. The signs are $+-$, $++$, $++$, $-+$, $-+$, $--$, $+-$.

19. $-\frac{\sqrt{2+\sqrt{2}}}{2}$. **20.** $-\{1+\sqrt{2}+\sqrt{2}\sqrt{2+\sqrt{2}}\}$.

EXAMPLES ON CHAPTER XIII, PAGE 58.

1. 2, 6, $\frac{1}{3}$, $\frac{3}{2}$, -12. **2.** 2, 0, 3, 2, -1, 1. **3.** 32.

4. ·2593838, $\bar{1}$·7781513, $\bar{3}$·1760913, ·6074552, ·535294, ·3714665, $\bar{1}$·8279869, ·19382, $\bar{2}$·732353, 3·8115752. **5.** 6. **6.** 1, $\frac{3}{2}$. **7.** 15.

8. 2·350248, 2·4593926, ·4771213, ·30103, ·845098, 2·807355, 1·747871, ·7156820. **9.** 6414·5, ·0064145. **10.** 1·6532126.

EXAMPLES ON CHAPTER XIV, PAGE 61.

1. 1·1355890, 136·6486. **2.** 2·2259904, ·001682653.

3. $\bar{3}$·1234400. **4.** 22·33292. **5.** 46016·78. **6.** 1·628895.

7. 2·441406. **8.** 22·98439. **9.** 8·465369. **10.** ·4409239.

11. 2·154435. **12.** 2·425805. **15.** 9·5407836, 9·5406301.

16. 9·7970899. **17.** 9·8993212. **18.** $32^\circ\, 0'\, 3''$.

19. 9·7825149, 9·7827428, $31^\circ\, 12'\, 22''$.

20. $L\sin 30^\circ = 9·69897$, $L\cos 30^\circ = 9·9375307$, 9·6990575, 9·9375015, 9·761556, 10·238444, 10·10624985, 10·3009425.

21. 9·9376764.

EXAMPLES ON CHAPTER XV, PAGES 68—70.

1. $A=45^\circ$, $B=45^\circ$, $c=90\sqrt{2}$. **2.** $A=30^\circ$, $B=60^\circ$, $c=6$.

3. $A=30^\circ$, $B=60^\circ$, $b=7\sqrt{3}$. **4.** $B=30^\circ$, $b=5\sqrt{3}$, $c=10\sqrt{3}$.

5. $B=60^{0}$, $a=6$, $b=6\sqrt{3}$. 6. $\sin B=\frac{4}{5}$, $b=4$, $c=5$.

8. 150 ft., 75 ft. 9. 1436·8 yards, 1387·6 yards nearly.

10. $\frac{3\sqrt{21}}{14}$, $\frac{\sqrt{21}}{14}$, $\sqrt{7}$. 11. $c=\sqrt{13}a$, $\sin A=\frac{1}{\sqrt{26}}$, $\sin B=\frac{3}{\sqrt{13}}$.

12. $c=10\sqrt{3}$, $A=90^{0}$, $B=30^{0}$. 14. No, it is right-angled.

15. $\cot\frac{C}{2}=\tan\frac{A+B}{2}=\frac{a+b}{a-b}\tan\frac{D}{2}$, this determines the angles: also $c=a\cos B+b\cos A$.

16. $a=s-\frac{S}{s}\cot\frac{A}{2}$; b and c are known since $b+c=2s-a$, $bc\sin A=2S$; $\sin B=\frac{b}{a}\sin A$, $B+C=180^{0}-A$.

17. $\frac{7}{8}$, $\frac{11}{16}$, $-\frac{1}{4}$, $\frac{\sqrt{15}}{6}$. 18. $\frac{13}{14}$, $\frac{15\sqrt{3}}{4}$.

19. 135^{0}, 30^{0}, 15^{0}. 20. $\frac{1}{\sqrt{21}}$, $\frac{\sqrt{21}}{3}$.

21. $c=7·6158$, $A=22^{0}\,46'\,12''$, $B=67^{0}\,13'\,48''$.

22. 73·202. 23. $41^{0}\,17'\,8''$. 24. 1836·078.

25. 700. 26. 1173·29. 27. 22·41479.

28. $b=28·865$, $c=34·4251$. 29. $B=107^{0}\,27'\,34''$, $C=18^{0}\,32'\,26''$.

30. $A=72^{0}\,12'\,59''$, $B=47^{0}\,47'\,1''$.

31. $A=68^{0}\,25'\,7''$, $B=37^{0}\,14'\,53''$.

32. $A=56^{0}\,2'\,12''$, $B=63^{0}\,57'\,48''$, $\log c=1·0559031$.

33. $B=70^{0}\,0'\,57''$, $C=59^{0}\,59'\,3''$, or $B=109^{0}\,59'\,3''$, $C=20^{0}\,0'\,57''$.

34. $A=72^{0}\,4'\,48''$, $B=41^{0}\,56'\,12''$, $b=17·5596$;
or $A=107^{0}\,55'\,12''$, $B=6^{0}\,5'\,48''$, $b=2·790537$.

35. (i) $B=60^{0}$, $C=90^{0}$, $b=125\sqrt{3}$; (ii) $B=111^{0}\,19'$, $C=38^{0}\,41'$, $b=372·502$; or $B=8^{0}\,41'$, $C=141^{0}\,19'$, $b=61·654$.

36. $22^{0}\,37'\,11''$, $67^{0}\,22'\,49''$, 90^{0}. 37. $63^{0}\,30'\,58''$.

38. $73^{0}\,23'\,54''$. 39. $41^{0}\,24'\,35''$. 40. $23^{0}\,45'\,27''$.

EXAMPLES ON CHAPTER XVI, PAGE 77.

1. $\frac{\pi}{12}$, $\frac{3\pi}{4}$, $\frac{601\pi}{3000}$, $\frac{12531\pi}{20000}$, $\frac{\pi}{80000}$, $\frac{\pi^2}{180}$, $\frac{\pi^2}{600}$.

2. 36^{0}, 300^{0}, $57^{0}\,17'\,44''·8$, $59^{0}\,21'\,56''·2$, 18^{0}.

3. $\frac{3}{8}\pi$. 5. $\frac{\pi}{5}$, $\frac{\pi}{3}$, $\frac{7\pi}{15}$. 6. $23^{0}\,52'\,14''$ nearly.

7. 63°, 80°, $\frac{\pi}{4}$. 8. $\frac{10\pi^2}{19\pi+1800}$, $\frac{9\pi^2}{19\pi+1800}$, $\frac{1800\pi}{19\pi+1800}$.

9. 12. 10. $\frac{\pi}{7}$, $\frac{6\pi}{7}$, 14. 17. 820, $88\frac{2}{3}$ miles an hour.

18. $84\frac{22}{337}$. 19. $51\frac{6}{11}$.

20. 880 sq. ft., $470\frac{5}{7}$ cubic feet. 21. $8\frac{1}{2}$ inches.

22. 18 ft. 9 in. 24. $1\frac{11}{42}$ miles. 25. $69^\circ 18'$.

26. 1680 yards. 27. $1382\frac{6}{7}$. 28. 154 yards.

29. $1\frac{4}{11}$ yards. 30. $5^\circ 40' 12''$. 34. $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)r^2$.

35. $\left(\sqrt{3}-\frac{\pi}{2}\right)r^2$. 36. $\frac{21}{55}^\circ$, $\frac{21}{275}^\circ$. 37. $\frac{11}{15}$ of an inch.

EXAMPLES ON CHAPTER XVII, PAGE 81.

1. $(4n\pm1)\frac{\pi}{4}$, $(6n\pm1)\frac{\pi}{6}$, $(2n+1)\frac{\pi}{2}$, $(3n\pm1)\frac{\pi}{3}$, $(6n\pm1)\frac{\pi}{6}$, $(6n\pm1)\frac{\pi}{6}$.

2. $\{6n+(-1)^n\}\frac{\pi}{6}$, $n\pi-(-1)^n\frac{\pi}{3}$, $(4n+1)\frac{\pi}{2}$, $(4n-1)\frac{\pi}{2}$, $n\pi-(-1)^n\frac{\pi}{10}$, $\{4n+(-1)^n\}\frac{\pi}{4}$.

3. $(6n\pm1)\frac{\pi}{3}$, $(12n\pm5)\frac{\pi}{6}$, $n\pi$, $(2n+1)\frac{\pi}{2}$, $(24n\pm1)\frac{\pi}{12}$, $(8n\pm3)\frac{\pi}{4}$.

4. $(4n+1)\frac{\pi}{4}$, $(4n+3)\frac{\pi}{4}$, $(3n+1)\frac{\pi}{3}$, $(6n-1)\frac{\pi}{6}$, $(12n+5)\frac{\pi}{12}$, $(12n-1)\frac{\pi}{12}$.

7. 3°, 15°, 39°, 51°, 75°, 87°. 8. 95°, 115°, 125°, 145°, 155°, 175°.

9. 189°, 225°, 261°. 10. $n\pi\pm\frac{\pi}{6}$. 11. $(8n+1)\frac{\pi}{4}\pm\frac{\pi}{3}$.

12. $\theta=n\pi$, or $2n\pi\pm\frac{\pi}{3}$. 13. $n\pi$, or α, where $\cos\alpha=\frac{\sqrt{5}-1}{2}$.

14. $\frac{n\pi}{2}$, or $(2n+1)\pi\pm\frac{\pi}{3}$. 15. $6\theta=(2n+1)\pi$, or $(2n+1)\frac{\pi}{2}$.

16. $(2n+1)\frac{\pi}{2}$. 17. $\frac{n\pi}{2}$. 18. $n\pi+\frac{\pi}{4}$, or $n\pi+\tan^{-1}3$.

19. $\theta=(4n\pm1)\frac{\pi}{4}$, or $\cos\theta=\frac{1}{3}$. 20. $2\pm\sqrt{5}$.

21. $\theta=2n\pi$, or $(2n+1)\frac{\pi}{6}$. 22. $2n\pi\pm\frac{\pi}{3}$.

23. $(4n-1)\frac{\pi}{2}$, or $(4n+1)\frac{\pi}{10}$. **24.** $n\pi$, or $(4n+1)\frac{\pi}{4}$.

25. $(3n\pm 1)\frac{\pi}{3}$. **26.** $2n\pi+\frac{\pi}{3}$.

27. $\tan x=\frac{3}{2}$, or $\cos x=\frac{2}{3}$, whence $x=n\pi+\tan^{-1}\frac{3}{2}$, or $2n\pi\pm\cos^{-1}\frac{2}{3}$.

EXAMPLES ON CHAPTER XVIII, PAGE 87.

3. 30^0. **5.** $\frac{1}{400}$, $\frac{1}{200}$, $\frac{1}{\sqrt{160001}}$, $\frac{1}{\sqrt{40001}}$.

6. $\tan^{-1}\sqrt{2}$. **7.** 170·23 feet nearly. **8.** $\dfrac{\sqrt{2}a}{(\cot^2\alpha+\cot^2\beta-2\cot^2\gamma)^{\frac{1}{2}}}$.

9. $40\sqrt{2}(\sqrt{3}-1)$. **11.** $\frac{c}{2}\tan 2a$. **12.** $c\sin 2a$. **13.** 50, 70·71.

14. $\dfrac{\sin\alpha\sin\beta}{\{\sin(\alpha+\beta)\sin(\alpha-\beta)\}^{\frac{1}{2}}}$, $\dfrac{\sin\beta}{\{\sin(\alpha+\beta)\sin(\alpha-\beta)\}^{\frac{1}{2}}}$ in miles.

16. 80 yards, 36 yards. **17.** 1954·7. **18.** 83·23129 feet.

22. 1034·13 feet. **23.** $200(\sqrt{3}-1)$, or 146·4 feet.

24. $50\sqrt{2}\sin 54^0$, or 114·41 feet. **25.** 18 miles.

26. $66\frac{2}{3}$ feet. **27.** 9 miles. **28.** 88 ft. 2 in.

29. $2^0\ 33'$ nearly; $2\frac{1742}{3969}$ of a mile. **30.** 3920 miles nearly.

MISCELLANEOUS EXAMPLES.

1. $70\frac{10}{77}$, $63\frac{9}{77}$, $46\frac{58}{77}$.

2. $2\sin 30^0-\tan 17^0\cot 17^0+\sec 34^0$, or $\sec 34^0$.

6. $\bar{1}\cdot 9739525$. **7.** $\frac{2\sqrt{3}}{3}p^2$. **8.** $x^2+y^2=a^2+b^2$.

9. $n\pi$, or $(2n+1)\frac{\pi}{8}$. **11.** $27^g\ 82'\ 25''$, $\frac{18029\pi}{129600}$.

12. $2^0\ 5'\ 54{\cdot}1''$ nearly. **13.** 1, or $\frac{1}{2}$. **15.** $\frac{2}{3ab+2}$, $\frac{3ab+2}{3a}$.

17. $x^2+y^2=2a^2$. **19.** $n\pi\pm a$. **21.** $\frac{6\pi}{19}$, $\frac{\pi}{3}$, $\frac{20\pi}{57}$.

22. 262 ft. 6 in. **29.** $(3n\pm 1)\frac{\pi}{3}$. **31.** 48^0, 60^0, 72^0.

32. 360, 1440. **36.** 1·991226, ·544068.

39. $(6n \pm 1)\frac{\pi}{3}$, or $(2n+1)\pi$. **41.** 18°, 72°. **42.** 1 mile per minute.

43. $\dfrac{m^2+n^2}{(m^4-12m^2n^2+n^4)^{\frac{1}{2}}}$. **48.** $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{m}+\dfrac{1}{n}$.

49. $n\pi$, or $(4n \pm 1)\frac{\pi}{4}$. **51.** $\frac{1}{8}$ of the unit of circular measurement.

61. 10°. **62.** $\dfrac{a^2}{2}$. **69.** $(4n+1)\frac{\pi}{4}$.

71. 45°, 63°, 72°, 50^g, 70^g, 80^g, $\frac{\pi}{4}$, $\frac{7\pi}{20}$, $\frac{2\pi}{5}$.

72. $(\pi-2)r^2 . (4-\pi)r^2$. **73.** $\dfrac{\sqrt{2+\sqrt{2}}}{2}$. **79.** $(6n+1)\frac{\pi}{3}$.

80. 6. **81.** 128°. **82.** $3\sqrt{3}-\pi : 3\sqrt{3}$.

89. $\dfrac{2n+1}{5}\pi$, $\dfrac{n\pi}{4}$, or $\dfrac{2n\pi}{3}$. **90.** 3960 miles nearly.

91. $\dfrac{12}{\pi}$. **96.** ·54. **98.** $y=mx$. **100.** 4050 miles nearly.

102. $\dfrac{11\pi}{24}$. **103.** $1+\dfrac{1}{2n(n+1)}$, $\dfrac{2n^2}{2n^2+2n+1}$.

109. $3(\sqrt{3}+1)$ feet. **111.** 10. **112.** 239520 miles.

115. 90°, 30°, 60°, $\sqrt{3}b$. **116.** $\dfrac{3\sqrt{51}}{22}$. **117.** $\dfrac{R^2}{2}\sin 2A \sin 2B \sin 2C$.

APPENDIX.

CAMBRIDGE UNIVERSITY PREVIOUS EXAMINATION PAPERS.

December 10, 1879. 1—3½ P.M. A.

1. DEFINE 1°. Assuming that $\frac{22}{7}$ is the circular measure of two right angles, express the angle A° in circular measure.

Find the number of degrees in the angle whose circular measure is $\cdot 1$.

2. Define the sine, secant, and cotangent of an angle, and express any two of these ratios in terms of the third.

Find the trigonometrical ratios of the angle whose cosine is $\frac{3}{5}$.

3. Prove that

$$(1)\quad \cos(180^{\circ}+A)=\cos(180^{\circ}-A),$$

$$(2)\quad \tan(90^{\circ}+A)=\cot(180^{\circ}-A).$$

4. Express the cosine of the difference of two angles in terms of the sines and cosines of these angles.

Prove that

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}.$$

5. Prove the formulæ:

$$(1)\quad \cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}.$$

$$(2)\quad \sin\frac{x}{2}+\cos\frac{x}{2}=\pm\sqrt{1+\sin x}.$$

$$(3)\quad \sin x\,(2\cos x-1)=2\sin\frac{x}{2}\cos\frac{3x}{2}.$$

6. Trace the changes in sign and magnitude of

$$\frac{2\sin\theta-\sin 2\theta}{2\sin\theta+\sin 2\theta},\ \text{as } \theta \text{ changes from 0 to } 2\pi.$$

7. Express the cosine of any angle of a triangle in terms of the sides. If the angle opposite the side a be 60°, and if b, c be the remaining sides of the triangle, prove that

$$(a+b+c)(b+c-a)=3bc.$$

8. Solve a triangle, having given the three sides.

Given $A=36^\circ$, $B=72^\circ$, and $a=1$,
solve the triangle.

9. The sides of a triangle are 2, 3, 4; find the least angle, having given

$$\begin{aligned} \log 2 &= \cdot 30103, \\ \log 3 &= \cdot 4771213, \\ \text{L} \tan 14^\circ\, 28' &= 9{\cdot}4116146, \\ \text{L} \tan 14^\circ\, 29' &= 9{\cdot}4121366. \end{aligned}$$

December 10, 1879. 1—3½ P.M. B.

1. Explain what is meant by the circular measure of an angle. Assuming $\frac{22}{7}$ to be the circular measure of two right angles, express in degrees the angle whose circular measure is θ.

Find the number of degrees in the angle whose circular measure is $\frac{1}{3}$.

2. Define the cosine, cosecant, and tangent of an angle, and express any two of these ratios in terms of the third.

Find all the trigonometrical ratios of the angle whose sine is $\frac{4}{5}$.

3. Prove that

(1) $\sin(180^\circ - A) = \sin A$,

(2) $\cot(90^\circ + A) = \tan(180^\circ - A)$.

4. Express the cosine of the sum of two angles in terms of the sines and cosines of these angles.

Prove that

$$\tan^{-1}x - \tan^{-1}y = \frac{x-y}{1+xy}.$$

5. Prove the formulæ:

(1) $\cos y - \cos x = 2 \sin \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$.

(2) $\sin \dfrac{x}{2} - \cos \dfrac{x}{2} = \pm \sqrt{1 - \sin x}$.

(3) $\sin x\,(2\cos x + 1) = 2 \cos \dfrac{x}{2} \sin \dfrac{3x}{2}$.

6. Trace the changes in sign and magnitude of

$$\frac{\sin\theta + 2\sin\frac{\theta}{2}}{\sin\theta - 2\sin\frac{\theta}{2}}, \text{ as } \theta \text{ changes from 0 to } 2\pi.$$

7. Prove that in any triangle the sides are proportional to the sines of the opposite angles.

If $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3 \sin A \sin B$,
then $C = 60^\circ$.

8. Solve a triangle, having given two angles and a side.

Given $$A=18^\circ,\ B=144^\circ,\ \text{and } b=1,$$
solve the triangle.

9. The sides of a triangle are 2, 3, 4; find the greatest angle, having given

$$\log 2 = \cdot 30103,$$
$$\log 3 = \cdot 4771213,$$
$$\text{L} \tan 52^\circ\ 15' = 10\cdot 1111004,$$
$$\text{L} \tan 52^\circ\ 14' = 10\cdot 1108395.$$

June 11, 1880. 1—3½ P.M. A.

1. Give the trigonometrical definition of an angle.

What angle does the minute hand of a clock describe between 12 o'clock and 20 minutes to 4?

2. Define the sine of an angle, and trace the change in the value of the sine as the angle increases from 90° to 270°.

3. Express the cosine and tangent of an angle in terms of the sine.

The angle A is greater than 90° but less than 180°, and $\sin A = \frac{1}{3}$; find $\cos A$.

4. Find $\sin 60^\circ$ and $\tan 135^\circ$.

5. Prove geometrically
$$\sin(A-B) = \sin A \cos B - \cos A \sin B,$$
A and B being positive angles less than 90°.

Shew that

$$(1)\quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

$$(2)\quad \sin 2A = \frac{2\tan A}{1+\tan^2 A}.$$

6. Shew that, if $$A+B+C=180^\circ,$$
$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

7. Find an expression for all the values of θ for which
$$\cos\theta + \cos 2\theta = 0.$$

8. Shew that in a triangle the sides are proportional to the sines of the opposite angles.

If in a triangle $a\cos A = b\cos B$, the triangle will be either isosceles or right-angled.

9. If two sides of a triangle be given, and the angle opposite to the shorter side, shew how to find the other side and the other angles.

Example: the sides are 1 foot and $\sqrt{3}$ feet respectively, and the angle opposite to the shorter side is 30^0.

June 11, 1880. 1—3½ P.M. B.

1. Distinguish between Euclid's definition of an angle and the trigonometrical definition.

What angle does the minute hand of a clock describe between half-past four and a quarter past six?

2. Define the cosine of an angle, and trace the change in the value of the cosine as the angle increases from 180^0 to 360^0.

3. Express the sine and the cosine of an angle in terms of the tangent.

The angle A is greater than 180^0, but less than 270^0, and $\tan A = \frac{1}{2}$; find $\sin A$.

4. Find $\cos 30^0$ and $\cot 135^0$.

5. Prove geometrically

$$\cos(A - B) = \cos A \cos B + \sin A \sin B,$$

A and B being both positive angles less than 90^0.

Shew that

$$(1)\quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

$$(2)\quad \sin 2A = \frac{2\cot A}{1 + \cot^2 A}.$$

6. Shew that, if $$A + B + C = 90^0,$$

$$\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C.$$

7. Find an expression for all the values of θ for which

$$\sin\theta + \sin 2\theta = 0.$$

8. Express the cosine of an angle of a triangle in terms of the sides.

If in a triangle $b\cos A = a\cos B$, shew that the triangle is isosceles.

9. If two sides of a triangle be given, and the angle opposite to one of them, shew how to find the other side and the other angles.

Example: the sides are 1 foot and $\sqrt{2}$ feet respectively, and the angle opposite to the shorter side is 30^0.

June 10, 1881. 1—3½ P.M. A.

1. Explain the measurement of angles by grades, &c.

Which is greater, 76^g or the angle whose circular measure is 1·2?

2. Define the sine, tangent, and secant of an angle, and prove from the definitions that $\tan A = \sin A \sec A$.

Having given $\sec A = 1\frac{1}{4}$, find $\sin A$ and $\tan A$.

3. Shew that when A is greater than B, and both are acute angles,

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

Also shew that

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2},$$

$$\sin^2 A - \sin^2 B = \sin(A+B)\cos(A-B).$$

4. Determine geometrically $\cos 30^\circ$ and $\cos 45^\circ$.

If $\sin A$ be the arithmetic mean between $\sin B$ and $\cos B$,

$$\cos 2A = \cos^2(B+45^\circ).$$

5. Establish the following relations:

(1) $\tan^2 A - \sin^2 A = \tan^2 A \sin^2 A$.

(2) $\cot A - \cot 2A = \operatorname{cosec} 2A$.

(3) $\dfrac{\sin(x+3y)+\sin(3x+y)}{\sin 2x + \sin 2y} = 2\cos(x+y)$.

6. Shew that for certain values of the angles

$$2\sin\frac{A}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

Is the formula true when $A = 240^\circ$; if not how must it be modified?

7. In any triangle the sides are proportional to the sines of the angles opposite to them.

8. Having given two angles of a triangle and one of the sides, shew how to find the other sides and the area of the triangle.

One side of a triangular lawn is 102 feet long, its inclinations to the other sides being 70° 30′, 78° 10′ respectively; determine the other sides and area.

$L\sin 70^\circ 30' = 9·974$, $L\sin 78^\circ 10' = 9·990$, $L\sin 31^\circ 20' = 9·716$.
$\log 102 = 2·009$, $\log 185 = 2·267$, $\log 192 = 2·283$,
$\log 2 = ·301$, $\log 9234 = 3·965$.

June 10, 1881. 1—3½ P.M. B.

1. Explain the measurement of angles by degrees, &c.

Which is greater, 126° or the angle whose circular measure is 2·3?

2. Define the cosine, cotangent and cosecant of an angle; and from the definitions prove that $\cot A = \cos A \operatorname{cosec} A$.

Having given $\cot A = \frac{3}{4}$, determine $\cos A$ and $\operatorname{cosec} A$.

3. Shew that when A is greater than B and both are acute angles

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Also shew that

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

$$\cos^2 B - \cos^2 A = \sin(A+B)\sin(A-B).$$

4. Determine geometrically $\tan 30^\circ$ and $\sin 45^\circ$.

If $\sin B$ be the geometric mean between $\sin A$ and $\cos A$,

$$\cos 2B = 2\cos^2(A+45^\circ).$$

5. Establish the following relations:

(1) $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$.

(2) $\tan A + \cot 2A = \operatorname{cosec} 2A$.

(3) $\dfrac{\cos(x-3y) - \cos(3x-y)}{\sin 2x + \sin 2y} = 2\sin(x-y)$.

6. Shew that for certain values of the angles

$$2\cos\frac{A}{2} = \sqrt{1+\sin A} + \sqrt{1-\sin A}.$$

Is the formula true when $A = 300^\circ$; if not how must it be modified?

7. Prove that in any triangle $a^2 = b^2 + c^2 - 2bc\cos A$.

8. Having given two sides of a triangle and the included angle, shew how to solve the triangle and find its area.

The sides of a triangular lawn are 102, 135 and 192 feet in length, the smallest angle being approximately $31^\circ\, 20'$, find its other angles and its area,

$\log 102 = 2{\cdot}009$, $\log 135 = 2{\cdot}130$, $\log 192 = 2{\cdot}283$, $\log 2 = {\cdot}301$,
$\log 9234 = 3{\cdot}965$, $L\sin 31^\circ 20' = 9{\cdot}716$, $L\sin 70^\circ 30' = 9{\cdot}974$, $L\sin 78^\circ 10' = 9{\cdot}990$.

EXAMINATION FOR ADMISSION TO THE ROYAL MILITARY ACADEMY, WOOLWICH (Obligatory).

June 25, 1881. 10—1.

1. Assuming that the ratio of the circumference of a circle to the diameter is 3·14156, find to four places of decimals the value in degrees of the ordinary unit of circular measure.

If the radius of a circle is 4000 miles, determine the number of miles in an arc which subtends an angle at its centre whose circular measure is $\frac{\pi}{6}$.

2. Give a definition of the sine of an angle which will apply to angles of any magnitude. Carefully prove that whatever be the magnitude of A,

$$\sin(90^\circ + A) = \cos A, \text{ and } \cos(90^\circ + A) = -\sin A.$$

3. If A and B be each less than 90°, but their sum greater, draw the appropriate figure, and give a geometrical proof that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Deduce the expression for $\sin(A+B)$ and shew that

$$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1.$$

4. Prove that

(1) $\tan A = \dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A}$.

(2) $\dfrac{\cos(A+45^\circ)}{\cos(A-45^\circ)} = \sec 2A - \tan 2A$.

(3) $\cot^{-1}\frac{1}{3} = \cot^{-1} 3 + \cot^{-1}\frac{3}{4}$.

5. Find $\sin 150^\circ$, $\sin 225^\circ$, $\sin 15^\circ$, $\sin 37^\circ 30'$ without reducing quadratic surds.

6. Obtain an expression for all the angles whose cosine is equal to a given value.

If $3\cos^2\theta + 2\sqrt{3}\cos\theta = 5\frac{1}{4}$, find the general value of θ.

Explain why it is that exactly the same series of angles are given by the two equations:

$$\theta + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6}; \text{ and } \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

7. Find expressions for the radii of the inscribed and escribed circles of a triangle, each expression involving one side and functions of the halves of the angles of the triangle.

If r, r_1, r_2, r_3 be the radii, and a, b, c the sides, prove that

$$\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}.$$

8. Shew that in any triangle

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Obtain an expression for the area in terms of the sides, and if the lengths of the sides be 242, 1212, and 1450 yards, shew that the area is 6 acres.

9. Assuming the formula

$$\log\sin(\theta + h) - \log\sin\theta = \mu h \cot\theta - \frac{\mu h^2}{2}\operatorname{cosec}^2\theta;$$

where h is the circular measure of a small angle, and μ the modulus of the logarithms, and powers of h above the second are neglected: explain when the principle of proportional parts will fail in its application to tables of logarithms of sines.

If L sin $15^0 30'$ be $9{\cdot}426899$ and the difference for $1'$ be $\cdot000455$, find accurately L sin $15^0 30' 36''$ and the angle whose tabular logarithmic cosine is equal to $9{\cdot}427263$.

Given $\log_{10} 2 = \cdot301030$ and $\log_{10} 3 = \cdot477121$, find $\log_{10} 3\frac{6}{25}$.

10. If in any triangle b, c, A be given, prove that

$$\tan\tfrac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2}:$$

hence obtain the formula $\tan\frac{1}{2}(B-C) = \tan^2\phi\cot\dfrac{A}{2}$, and explain the advantage of such a formula.

If $b : c :: 11 : 10$, and $A = 35^0 25'$, apply the above to find B and C, having given $\log 1{\cdot}1 = \cdot041393$,

$$\text{L}\cos 24^0 37' 12'' = 9{\cdot}958607 : \text{L}\tan 12^0 18' 36'' = 9{\cdot}338891,$$
$$\text{L}\cot 17^0 42' 30'' = 10{\cdot}4958 : \text{L}\tan 8^0 28' 56{\cdot}5'' = 9{\cdot}173582.$$

11. From a house on one side of a street observations are made of the angle subtended by the height of the opposite house: first from the level of the street, in which case the angle is $\tan^{-1}(3)$; and afterwards from two windows one above the other, from each of which the angle is found to be $\tan^{-1}(-3)$. The height of the house opposite being 60 feet, find the height of each of the two windows above the street.

EXAMINATION FOR ADMISSION TO THE ROYAL MILITARY ACADEMY, WOOLWICH (Obligatory).

November 26, 1881. 10—1.

1. Define the "circular measure" of an angle, and shew that, if with centre O any circle be described, cutting the bounding lines of an angle BOC in the points P, Q, the circular measure of the angle will be equal to $\dfrac{\text{arc } PQ}{OP}$.

Find the number of seconds in the angle subtended at the centre of a circle, whose radius is one mile, by an arc $5\frac{1}{2}$ inches long.

2. Give accurate definitions of the sine and cosine of an angle, and prove that

$$\cos A = \sin(90^0 + A) = -\cos(180^0 + A).$$

Shew that the sine will be algebraically less than the cosine for any angle between $(8n-3)\,45^0$ and $(8n+1)\,45^0$, where n is zero or any positive integer.

3. Find, geometrically, expressions for the sine and cosine of the sum of two angles in terms of the sines and cosines of the angles themselves.

The cosines of two angles of a triangle are $\frac{3}{5}$ and $\frac{12}{13}$ respectively. Find all the trigonometrical ratios of the third angle.

4. Express $\sin A$ and $\cos A$ in terms of $\tan\frac{A}{2}$.

Prove that $\tan\left(45^\circ - \frac{A}{2}\right) + \tan\left(45^\circ + \frac{A}{2}\right) = 2\sec A$.

5. Find the sine of 18°, and deduce the sine of 36°.

Two parallel chords of a circle, lying on the same side of the centre, subtend respectively 72° and 144° at the centre. Shew that the distance between the chords is half the radius of the circle.

6. Find all the angles which satisfy the equation $2\sin 2\theta = 3\tan\theta$.

7. Prove that:

(1) $\sec^6\theta - \tan^6\theta = 1 + 3\tan^2\theta\sec^2\theta$.

(2) $\text{vers}(270^\circ + A) . \text{vers}(270^\circ - A) = \cos^2 A$.

(3) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\cos A - 2\cos 3A + \cos 5A} = \dfrac{4\sin A - 3\operatorname{cosec} A}{4\cos A - 3\sec A}$.

8. Given $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$, find $\text{L}\sin 45^\circ$, $\text{L}\sec 30^\circ$. Also if $\text{L}\sin 15^\circ = 9\cdot 4129962$, what will be the value of $\text{L}\cos 15^\circ$?

9. Prove that in any triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

If the angles adjacent to the base of a triangle are $22\frac{1}{2}^\circ$ and $112\frac{1}{2}^\circ$, shew that the perpendicular altitude will be one-half the base.

10. Find an expression for the radius of the circle which touches one side of a triangle and the other two produced.

Shew that the sum of the radii of the two escribed circles of a triangle, which touch the side c produced, is equal to $c\cot\frac{C}{2}$.

11. Shew how to solve a triangle by means of logarithms when the three sides are given.

Find the least angle of the triangle whose sides are 24, 22, 14, having given $\text{L}\tan 17^\circ 33' = 9\cdot 500042$, diff. for $1' = \cdot 000439$.

12. A man walking along a straight road, which runs in a direction 30° East of North, notes when he is due South of a certain house. When he has walked a mile further, he observes that the house lies due West, and that a windmill on the opposite side of the road is N.E. of him. Three miles further on, he finds that he is due North of the windmill. Find the distance between the house and the windmill, and shew that the line joining them makes with the road an angle $\tan^{-1}\left(\frac{48 - 25\sqrt{3}}{11}\right)$.

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